Cup product

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Chapter 1

Cap product

In algebraic topology the **cap product** is a method of adjoining a chain of degree p with a cochain of degree q, such that $q \le p$, to form a composite chain of degree $p - q$. It was introduced by Eduard Čech in 1936, and independently by Hassler Whitney in 1938.

1.1 Definition

Let X be a topological space and R a coefficient ring. The cap product is a bilinear map on singular homology and cohomology

$$\frown : H_p(X;R) \times H^q(X;R) \to H_{p-q}(X;R).$$

defined by contracting a singular chain $\sigma : \Delta^p \to X$ with a singular cochain $\psi \in C^q(X;R)$, by the formula :

$$\sigma \frown \psi = \psi(\sigma|_{[v_0,\ldots,v_q]})\sigma|_{[v_q,\ldots,v_p]}.$$

Here, the notation $\sigma|_{[v_0,\ldots,v_q]}$ indicates the restriction of the simplicial map σ to its face spanned by the vectors of the base, see Simplex.

1.2 Interpretation

In analogy with the interpretation of the cup product in terms of the Künneth formula, we can explain the existence of the cap product by considering the composition

$$C_\bullet(X) \otimes C^\bullet(X) \overset{\Delta_* \otimes \mathrm{Id}}{\longrightarrow} C_\bullet(X) \otimes C_\bullet(X) \otimes C^\bullet(X) \overset{\mathrm{Id} \otimes \varepsilon}{\longrightarrow} C_\bullet(X)$$

in terms of the chain and cochain complexes of X , where we are taking tensor products of chain complexes, $\Delta \colon X \to X \times X$ is the diagonal map which induces the map Δ_* on the chain complex, and $\varepsilon \colon C_p(X) \otimes C^q(X) \to \mathbb{Z}$ is the evaluation map (always 0 except for $p = q$).

This composition then passes to the quotient to define the cap product $\frown \colon H_\bullet(X) \times H^\bullet(X) \to H_\bullet(X)$, and looking carefully at the above composition shows that it indeed takes the form of maps $\frown \colon H_p(X) \times H^q(X) \to H_{p-q}(X)$, which is always zero for $p < q$.

1.3 The slant product

The above discussion indicates that the same operation can be defined on cartesian products $X \times Y$ yielding a product

$\backslash : H_p(X;R) \otimes H^q(X \times Y;R) \to H^{q-p}(Y;R).$

In case $X = Y$, the two products are related by the diagonal map.

1.4 Equations

The boundary of a cap product is given by :

$$\partial(\sigma \frown \psi) = (-1)^q(\partial\sigma \frown \psi - \sigma \frown \delta\psi).$$

Given a map f the induced maps satisfy :

$$f_*(\sigma) \frown \psi = f_*(\sigma \frown f^*(\psi)).$$

The cap and cup product are related by :

$$\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma)$$

where

$\sigma : \Delta^{p+q} \to X$, $\psi \in C^q(X;R)$ and $\varphi \in C^p(X;R).$

An interesting consequence of the last equation is that it makes $H_*(X;R)$ into a right $H^*(X;R)-$ module.

1.5 See also

- cup product
- Poincaré duality
- singular homology
- homology theory

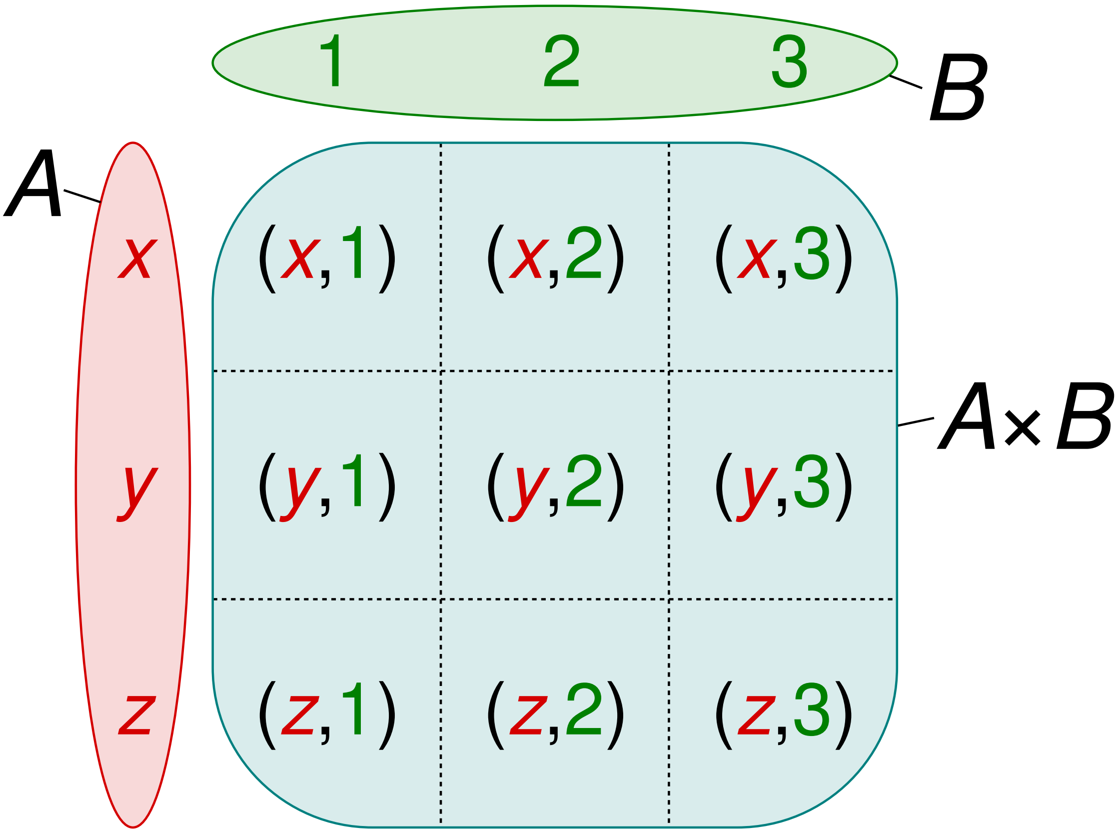
1.6 References

- Hatcher, A., *Algebraic Topology,* Cambridge University Press (2002) ISBN 0-521-79540-0. Detailed discussion of homology theories for simplicial complexes and manifolds, singular homology, etc.
- slant product in *nLab*

Chapter 2

Cartesian product

“Cartesian square” redirects here. For Cartesian squares in category theory, see Cartesian square (category theory).
In mathematics, a **Cartesian product** is a mathematical operation which returns a set (or **product set** or simply



Cartesian product $A \times B$ *of the sets* $A=\{x,y,z\}$ *and* $B=\{1,2,3\}$

product) from multiple sets. That is, for sets *A* and *B*, the Cartesian product *A* × *B* is the set of all ordered pairs (*a*, *b*) where *a* ∈ *A* and *b* ∈ *B*. Products can be specified using set-builder notation, e.g.

$$A \times B = \{\,(a,b) \mid a \in A \ \text{ and } \ b \in B\,\}.$$ [1]

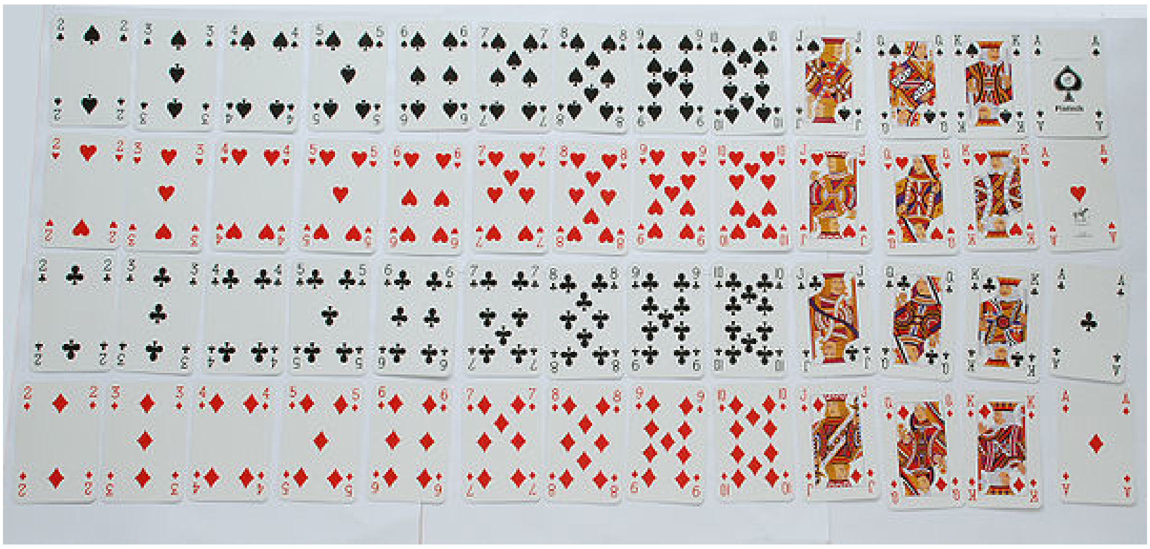
A table can be created by taking the Cartesian product of a set of rows and a set of columns. If the Cartesian product *rows* × *columns* is taken, the cells of the table contain ordered pairs of the form (row value, column value).

More generally, a Cartesian product of *n* sets, also known as an ***n*-fold Cartesian product**, can be represented by an array of *n* dimensions, where each element is an *n*-tuple. An ordered pair is a 2-tuple or **couple**.

The Cartesian product is named after René Descartes,[2] whose formulation of analytic geometry gave rise to the concept, which is further generalized in terms of direct product.

2.1 Examples

2.1.1 A deck of cards



Standard 52-card deck

An illustrative example is the standard 52-card deck. The standard playing card ranks {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} form a 13-element set. The card suits {♠, ♥, ♦, ♣} form a 4-element set. The Cartesian product of these sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards.

Ranks × *Suits* returns a set of the form {(A, ♠), (A, ♥), (A, ♦), (A, ♣), (K, ♠), ..., (3, ♣), (2, ♠), (2, ♥), (2, ♦), (2, ♣)}.

Suits × *Ranks* returns a set of the form {(♠, A), (♠, K), (♠, Q), (♠, J), (♠, 10), ..., (♣, 6), (♣, 5), (♣, 4), (♣, 3), (♣, 2)}.

Both sets are distinct, even disjoint.

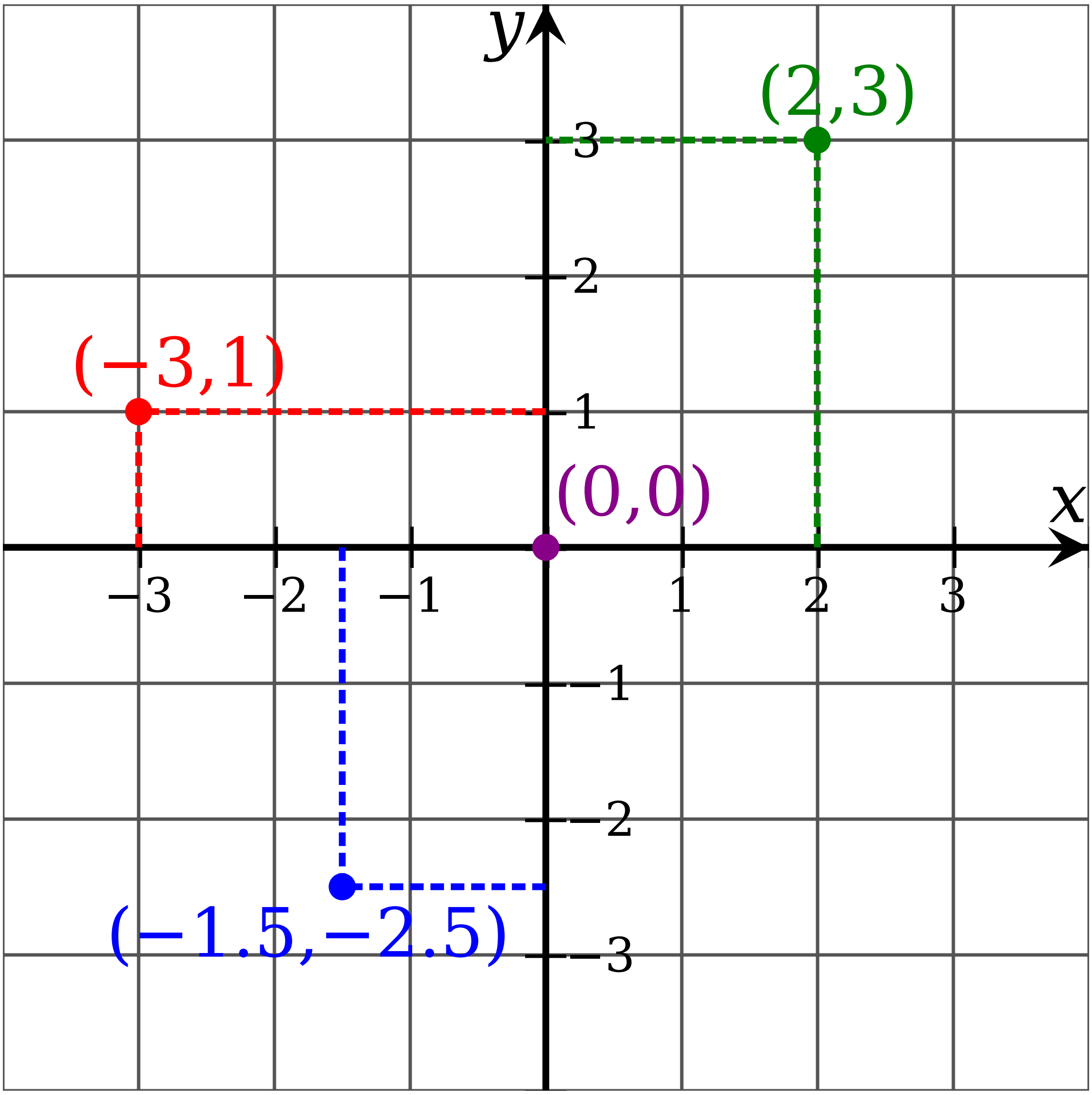
2.1.2 A two-dimensional coordinate system

The main historical example is the Cartesian plane in analytic geometry. In order to represent geometrical shapes in a numerical way and extract numerical information from shapes' numerical representations, René Descartes assigned to each point in the plane a pair of real numbers, called its coordinates. Usually, such a pair's first and second components are called its *x* and *y* coordinates, respectively; cf. picture. The set of all such pairs, i.e., the Cartesian product $\mathbb{R}\times\mathbb{R}$ with $\mathbb{R}$ denoting the real numbers, is thus assigned to the set of all points in the plane.

2.2 Most common implementation (set theory)

Main article: Implementation of mathematics in set theory

A formal definition of the Cartesian product from set-theoretical principles follows from a definition of ordered pair. The most common definition of ordered pairs, the Kuratowski definition, is $(x, y) = \{\{x\}, \{x, y\}\}$. Note that, under this definition, $X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y))$, where $\mathcal{P}$ represents the power set. Therefore, the existence of the Cartesian product of any two sets in ZFC follows from the axioms of pairing, union, power set, and specification.



Cartesian coordinates of example points

Since functions are usually defined as a special case of relations, and relations are usually defined as subsets of the Cartesian product, the definition of the two-set Cartesian product is necessarily prior to most other definitions.

2.2.1 Non-commutativity and non-associativity

Let A, B, C, and D be sets.

The Cartesian product $A \times B$ is not commutative,

$$A \times B \neq B \times A,$$

because the ordered pairs are reversed except if at least one of the following conditions is satisfied:[3]

- A is equal to B, or
- A or B is the empty set.

For example:

$A = \{1,2\}; B = \{3,4\}$

$A \times B = \{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$
$B \times A = \{3,4\} \times \{1,2\} = \{(3,1), (3,2), (4,1), (4,2)\}$

$A = B = \{1,2\}$

$A \times B = B \times A = \{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

$A = \{1,2\}; B = \varnothing$

$A \times B = \{1,2\} \times \varnothing = \varnothing$
$B \times A = \varnothing \times \{1,2\} = \varnothing$

Strictly speaking, the Cartesian product is not associative (unless one of the involved sets is empty).

$$(A \times B) \times C \neq A \times (B \times C)$$

If for example $A = \{1\}$, then $(A \times A) \times A = \{ ((1,1),1) \} \neq \{ (1,(1,1)) \} = A \times (A \times A)$.

2.2.2 Intersections, unions, and subsets

The Cartesian product behaves nicely with respect to intersections, cf. left picture.

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$ [4]

In most cases the above statement is not true if we replace intersection with union, cf. middle picture.

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

In fact, we have that:

$$(A \times C) \cup (B \times D) = [(A \setminus B) \times C] \cup [(A \cap B) \times (C \cup D)] \cup [(B \setminus A) \times D]$$

For the set difference we also have the following identity:

$$(A \times C) \setminus (B \times D) = [A \times (C \setminus D)] \cup [(A \setminus B) \times C]$$

Here are some rules demonstrating distributivity with other operators (cf. right picture):[3]

$$A \times (B \cap C) = (A \times B) \cap (A \times C),$$
$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C),$$
$$(A \times B)^c = (A^c \times B^c) \cup (A^c \times B) \cup (A \times B^c).$$ [4]

Other properties related with subsets are:

if $A \subseteq B$ then $A \times C \subseteq B \times C$,
both if $A, B \neq \emptyset$ then $A \times B \subseteq C \times D \iff A \subseteq C \wedge B \subseteq D.$ [5]

2.2.3 Cardinality

See also: Cardinal arithmetic

The cardinality of a set is the number of elements of the set. For example, defining two sets: A = {a, b} and B = {5, 6}. Both set A and set B consist of two elements each. Their Cartesian product, written as $A \times B$, results in a new set which has the following elements:

$A \times B$ = {(a,5), (a,6), (b,5), (b,6)}.

Each element of A is paired with each element of B. Each pair makes up one element of the output set. The number of values in each element of the resulting set is equal to the number of sets whose cartesian product is being taken; 2 in this case. The cardinality of the output set is equal to the product of the cardinalities of all the input sets. That is,

$|A \times B| = |A| \cdot |B|$.

Similarly

$|A \times B \times C| = |A| \cdot |B| \cdot |C|$

and so on.

The set $A \times B$ is infinite if either A or B is infinite and the other set is not the empty set.[6]

2.3 *n*-ary product

2.3.1 Cartesian power

The **Cartesian square** (or **binary Cartesian product**) of a set X is the Cartesian product $X^2 = X \times X$. An example is the 2-dimensional plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ where $\mathbf{R}$ is the set of real numbers: $\mathbf{R}^2$ is the set of all points (x,y) where x and y are real numbers (see the Cartesian coordinate system).

The **cartesian power** of a set X can be defined as:

$$X^n = \underbrace{X \times X \times \cdots \times X}_{n} = \{(x_1, \ldots, x_n) \mid x_i \in X \text{ all for } i = 1, \ldots, n\}.$$

An example of this is $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, with $\mathbf{R}$ again the set of real numbers, and more generally $\mathbf{R}^n$.

The n-ary cartesian power of a set X is isomorphic to the space of functions from an n-element set to X. As a special case, the 0-ary cartesian power of X may be taken to be a singleton set, corresponding to the empty function with codomain X.

2.3.2 Finite n-ary product

The Cartesian product can be generalized to the ***n*-ary Cartesian product** over n sets X_1, ..., Xn:

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_i \in X_i\}.$$

It is a set of n-tuples. If tuples are defined as nested ordered pairs, it can be identified to $(X_1 \times \ldots \times Xn{-}1) \times Xn$.

2.3.3 Infinite products

It is possible to define the Cartesian product of an arbitrary (possibly infinite) indexed family of sets. If *I* is any index set, and $\{X_i\}_{i\in I}$ is a family of sets indexed by *I*, then the Cartesian product of the sets in *X* is defined to be

$$\prod_{i\in I} X_i = \left\{ f : I \to \bigcup_{i\in I} X_i \;\middle|\; (\forall i)(f(i) \in X_i) \right\},$$

that is, the set of all functions defined on the index set such that the value of the function at a particular index *i* is an element of *Xi*. Even if each of the *Xi* is nonempty, the Cartesian product may be empty if the axiom of choice (which is equivalent to the statement that every such product is nonempty) is not assumed.

For each *j* in *I*, the function

$$\pi_j : \prod_{i\in I} X_i \to X_j,$$

defined by $\pi_j(f) = f(j)$ is called the ***j*****th projection map**.

An important case is when the index set is $\mathbb{N}$, the natural numbers: this Cartesian product is the set of all infinite sequences with the *i*th term in its corresponding set *Xi*. For example, each element of

$$\prod_{n=1}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \cdots$$

can be visualized as a vector with countably infinite real number components. This set is frequently denoted $\mathbb{R}^\omega$, or $\mathbb{R}^\mathbb{N}$.

The special case **Cartesian exponentiation** occurs when all the factors *Xi* involved in the product are the same set *X*. In this case,

$$\prod_{i\in I} X_i = \prod_{i\in I} X$$

is the set of all functions from *I* to *X*, and is frequently denoted X^I. This case is important in the study of cardinal exponentiation.

The definition of finite Cartesian products can be seen as a special case of the definition for infinite products. In this interpretation, an *n*-tuple can be viewed as a function on {1, 2, ..., *n*} that takes its value at *i* to be the *i*th element of the tuple (in some settings, this is taken as the very definition of an *n*-tuple).

2.4 Other forms

2.4.1 Abbreviated form

If several sets are being multiplied together, e.g. X_1, X_2, X_3, …, then some authors[7] choose to abbreviate the Cartesian product as simply ×*Xi*.

2.4.2 Cartesian product of functions

If *f* is a function from *A* to *B* and *g* is a function from *X* to *Y*, their **Cartesian product** *f* × *g* is a function from *A* × *X* to *B* × *Y* with

$$(f \times g)(a, b) = (f(a), g(b)).$$

This can be extended to tuples and infinite collections of functions. Note that this is different from the standard cartesian product of functions considered as sets.

2.5 Definitions outside of Set theory

2.5.1 Category theory

Although the Cartesian product is traditionally applied to sets, category theory provides a more general interpretation of the product of mathematical structures. This is distinct from, although related to, the notion of a Cartesian square in category theory, which is a generalization of the fiber product.

Exponentiation is the right adjoint of the Cartesian product; thus any category with a Cartesian product (and a final object) is a Cartesian closed category.

2.5.2 Graph theory

In graph theory the Cartesian product of two graphs G and H is the graph denoted by $G \times H$ whose vertex set is the (ordinary) Cartesian product $V(G) \times V(H)$ and such that two vertices (u,v) and (u',v') are adjacent in $G \times H$ if and only if $u = u'$ and v is adjacent with v' in H, *or* $v = v'$ and u is adjacent with u' in G. The Cartesian product of graphs is not a product in the sense of category theory. Instead, the categorical product is known as the tensor product of graphs.

2.6 See also

- Exponential object
- Binary relation
- Coproduct
- Empty product
- Product (category theory)
- Concatenation of sets is deceptively similar but different concept
- Product topology
- Finitary relation
- Ultraproduct
- Product type
- Euclidean space
- Orders on $\mathbf{R}^n$
- Join (SQL), § Cross join

2.7 References

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[2] cartesian. (2009). In Merriam-Webster Online Dictionary. Retrieved December 1, 2009, from http://www.merriam-webster.com/dictionary/cartesian

[3] Singh, S. (2009, August 27). *Cartesian product*. Retrieved from the Connexions Web site: http://cnx.org/content/m15207/1.5/

[4] CartesianProduct at PlanetMath.org.

[5] Cartesian Product of Subsets. (2011, February 15). *ProofWiki*. Retrieved 05:06, August 1, 2011 from https://proofwiki.org/w/index.php?title=Cartesian_Product_of_Subsets&oldid=45868

[6] Peter S. (1998). A Crash Course in the Mathematics Of Infinite Sets. *St. John's Review, 44*(2), 35–59. Retrieved August 1, 2011, from http://www.mathpath.org/concepts/infinity.htm

[7] Osborne, M., and Rubinstein, A., 1994. *A Course in Game Theory*. MIT Press.

2.8 External links

- Cartesian Product at ProvenMath
- Hazewinkel, Michiel, ed. (2001), "Direct product", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- How to find the Cartesian Product, Education Portal Academy

Chapter 3

Circular convolution

The **circular convolution**, also known as **cyclic convolution**, of two aperiodic functions (i.e. Schwartz functions) occurs when one of them is convolved in the normal way with a periodic summation of the other function. That situation arises in the context of the Circular convolution theorem. The identical operation can also be expressed in terms of the periodic summations of both functions, if the infinite integration interval is reduced to just one period. That situation arises in the context of the discrete-time Fourier transform (DTFT) and is also called **periodic convolution**. In particular, the DTFT of the product of two discrete sequences is the periodic convolution of the DTFTs of the individual sequences.[1]

Let x be a function with a well-defined periodic summation, xT, where:

$$x_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} x(t-kT) = \sum_{k=-\infty}^{\infty} x(t+kT).$$

If h is any other function for which the convolution $xT * h$ exists, then the convolution $xT * h$ is periodic and identical to**:**

$$\begin{aligned}(x_T * h)(t) \quad &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)\cdot x_T(t-\tau)\,d\tau \\ &\equiv \int_{t_o}^{t_o+T} h_T(\tau)\cdot x_T(t-\tau)\,d\tau,\end{aligned}$$ [2]

where t_o is an arbitrary parameter and hT is a periodic summation of h.

The second integral is called the **periodic convolution**[3][4] of functions xT and hT and is sometimes normalized by $1/T$.[5] When xT is expressed as the periodic summation of another function, x, the same operation may also be referred to as a **circular convolution**[4][6] of functions h and x.

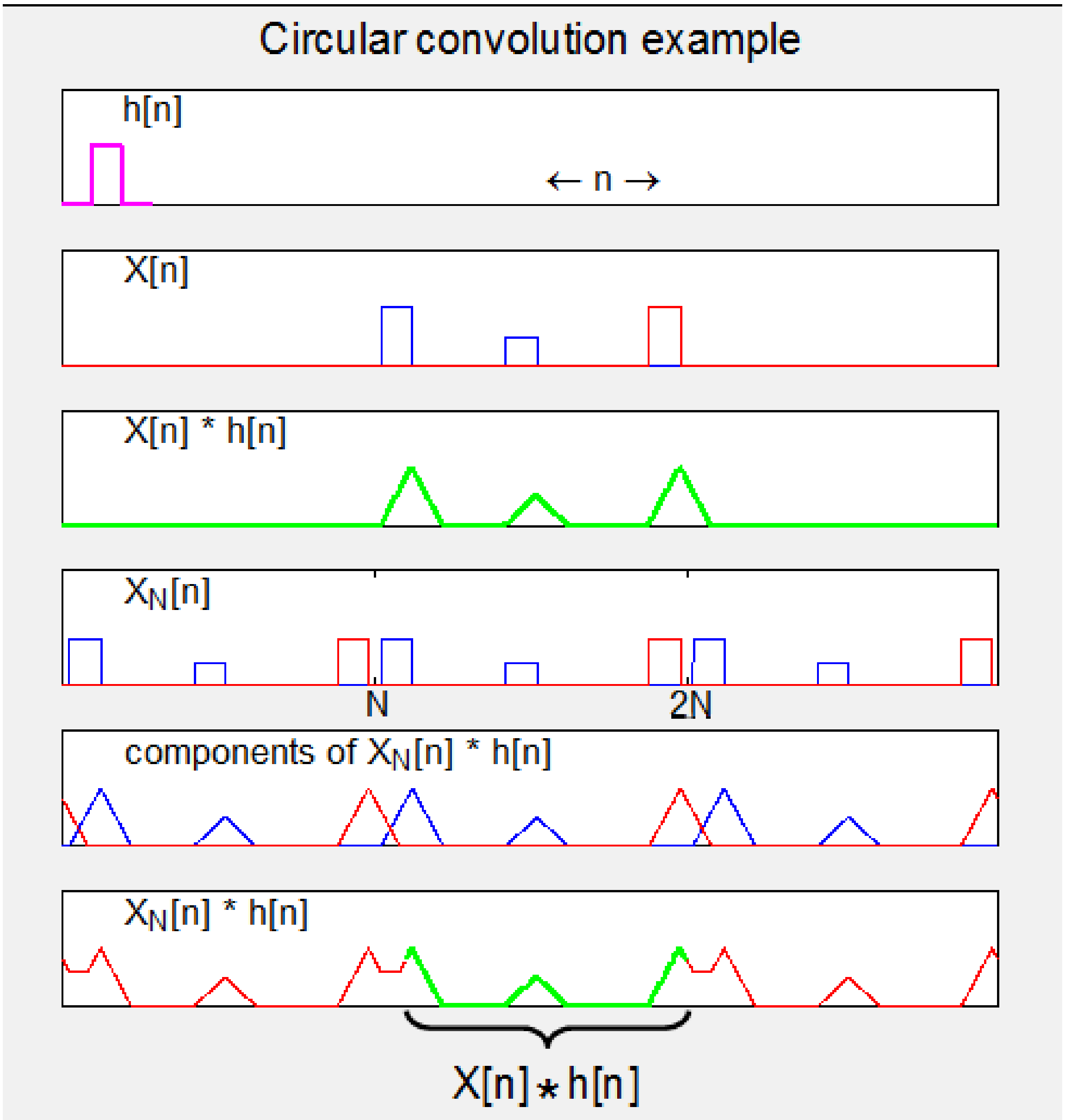
3.1 Discrete sequences

Similarly, for discrete sequences and period **N**, we can write the **circular convolution** of functions h and x as**:**

$$\begin{aligned}(x_N * h)[n] &\stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} h[m]\cdot x_N[n-m] \\ &= \sum_{m=-\infty}^{\infty} \left(h[m]\cdot \sum_{k=-\infty}^{\infty} x[n-m-kN] \right).\end{aligned}$$

For the special case that the non-zero extent of both x and h are $\leq N$, this is reducible to matrix multiplication where the kernel of the integral transform is a circulant matrix.

3.2 Example



A case of great practical interest is illustrated in the figure. The duration of the **x** sequence is **N** (or less), and the duration of the **h** sequence is significantly less. Then many of the values of the circular convolution are identical to values of **x∗h**, which is actually the desired result when the **h** sequence is a finite impulse response (FIR) filter. Furthermore, the circular convolution is very efficient to compute, using a fast Fourier transform (FFT) algorithm and the circular convolution theorem.

There are also methods for dealing with an **x** sequence that is longer than a practical value for **N**. The sequence is divided into segments (*blocks*) and processed piecewise. Then the filtered segments are carefully pieced back together. Edge effects are eliminated by overlapping either the input blocks or the output blocks. To help explain and compare the methods, we discuss them both in the context of an **h** sequence of length 201 and an FFT size of $N = 1024$.

Overlapping input blocks

This method uses a block size equal to the FFT size (1024). We describe it first in terms of normal or *linear* convolution. When a normal convolution is performed on each block, there are start-up and decay transients at the block edges, due to the filter *latency* (200-samples). Only 824 of the convolution outputs are unaffected by edge effects. The others are discarded, or simply not computed. That would cause gaps in the output if the input blocks are contiguous. The gaps are avoided by overlapping the input blocks by 200 samples. In a sense, 200 elements from each input block are "saved" and carried over to the next block. This method is referred to as **overlap-save**,[7] although the method we describe next requires a similar "save" with the output samples.

When the DFT or FFT is used, we don't have the option of not computing the affected samples, but the leading and trailing edge-effects are overlapped and added because of circular convolution. Consequently, the 1024-point inverse FFT (IFFT) output contains only 200 samples of edge effects (which are discarded) and the 824 unaffected samples (which are kept). To illustrate this, the fourth frame of the figure at right depicts a block that has been periodically (or "circularly") extended, and the fifth frame depicts the individual components of a linear convolution performed on the entire sequence. The edge effects are where the contributions from the extended blocks overlap the contributions from the original block. The last frame is the composite output, and the section colored green represents the unaffected portion.

Overlapping output blocks

This method is known as **overlap-add**.[8] In our example, it uses contiguous input blocks of size 824 and pads each one with 200 zero-valued samples. Then it overlaps and adds the 1024-element output blocks. Nothing is discarded, but 200 values of each output block must be "saved" for the addition with the next block. Both methods advance only 824 samples per 1024-point IFFT, but overlap-save avoids the initial zero-padding and final addition.

3.3 See also

- Discrete Hilbert transform
- Circulant matrix

3.4 Notes

[1] If a sequence, $x[n]$, represents samples of a continuous function, $x(t)$, with Fourier transform $X(f)$, its DTFT is a periodic summation of $X(f)$. (see Discrete-time_Fourier_transform#Relationship_to_sampling)

[2] Proof:

$$\int_{-\infty}^{\infty} h(\tau)\cdot x_T(t-\tau)\,d\tau$$

$$\begin{aligned}
&= \sum_{k=-\infty}^{\infty}\left[\int_{t_o+kT}^{t_o+(k+1)T} h(\tau)\cdot x_T(t-\tau)\,d\tau\right]\\
&\stackrel{\tau\rightarrow\tau+kT}{=} \sum_{k=-\infty}^{\infty}\left[\int_{t_o}^{t_o+T} h(\tau+kT)\cdot x_T(t-\tau-kT)\,d\tau\right]\\
&= \int_{t_o}^{t_o+T}\left[\sum_{k=-\infty}^{\infty} h(\tau+kT)\cdot \underbrace{x_T(t-\tau-kT)}_{X_T(t-\tau),\text{periodicity by}}\right]d\tau\\
&= \int_{t_o}^{t_o+T}\underbrace{\left[\sum_{k=-\infty}^{\infty} h(\tau+kT)\right]}_{\stackrel{\text{def}}{=}\, h_T(\tau)}\cdot x_T(t-\tau)\,d\tau \qquad (QED)
\end{aligned}$$

[3] Jeruchim 2000, pp 73-74.

[4] Udayashankara 2010, p 189.

[5] Oppenheim, pp 388-389

[6] Priemer 1991, pp 286-289.

[7] Rabiner 1975, pp 65–67.

[8] Rabiner 1975, pp 63–65.

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Chapter 4

Commutator

This article is about the mathematical concept. For the relation between canonical conjugate entities, see Canonical commutation relation. For the type of electrical switch, see Commutator (electric).

In mathematics, the **commutator** gives an indication of the extent to which a certain binary operation fails to be commutative. There are different definitions used in group theory and ring theory.

4.1 Group theory

The **commutator** of two elements, g and h, of a group G, is the element

$$[g, h] = g^{-1}h^{-1}gh.$$

It is equal to the group's identity if and only if g and h commute (i.e., if and only if $gh = hg$). The subgroup of G generated by all commutators is called the *derived group* or the *commutator subgroup* of G. Note that one must consider the subgroup generated by the set of commutators because in general the set of commutators is not closed under the group operation. Commutators are used to define nilpotent and solvable groups.

The above definition of the commutator is used by some group theorists, as well as throughout this article. However, many other group theorists define the commutator as

$$[g, h] = ghg^{-1}h^{-1}.$$ [1][2]

4.1.1 Identities (group theory)

Commutator identities are an important tool in group theory.[3] The expression a^x denotes the conjugate of a by x, defined as $x^{-1}a\,x$.

1. $x^y = x[x, y]$.
2. $[y, x] = [x, y]^{-1}$.
3. $[x, zy] = [x, y] \cdot [x, z]^y$ and $[xz, y] = [x, y]^z \cdot [z, y]$.
4. $[x, y^{-1}] = [y, x]^{y^{-1}}$ and $[x^{-1}, y] = [y, x]^{x^{-1}}$.
5. $[[x, y^{-1}], z]^y \cdot [[y, z^{-1}], x]^z \cdot [[z, x^{-1}], y]^x = 1$ and $[[x, y], z^x] \cdot [[z, x], y^z] \cdot [[y, z], x^y] = 1$.

Identity 5 is also known as the *Hall–Witt identity*. It is a group-theoretic analogue of the Jacobi identity for the ring-theoretic commutator (see next section).

N.B. The above definition of the conjugate of a by x is used by some group theorists.[4] Many other group theorists define the conjugate of a by x as xax^{-1}.[5] This is often written ${}^x a$. Similar identities hold for these conventions.

A wide range of identities are used that are true modulo certain subgroups. These can be particularly useful in the study of solvable groups and nilpotent groups. For instance, in any group second powers behave well,

$$(xy)^2 = x^2y^2[y,x][[y,x],y].$$

If the derived subgroup is central, then

$$(xy)^n = x^ny^n[y,x]^{\binom{n}{2}}.$$

4.2 Ring theory

The **commutator** of two elements a and b of a ring or an associative algebra is defined by

$$[a,b] = ab - ba.$$

It is zero if and only if a and b commute. In linear algebra, if two endomorphisms of a space are represented by commuting matrices with respect to one basis, then they are so represented with respect to every basis. By using the commutator as a Lie bracket, every associative algebra can be turned into a Lie algebra.

The **anticommutator** of two elements a and b of a ring or an associative algebra is defined by

$$\{a,b\} = ab + ba.$$

Sometimes the brackets $[\ ]_+$ are also used to denote anticommutators.[6] The anticommutator is used less often than the commutator, but can be used for example to define Clifford algebras, Jordan algebras and is utilised to derive the Dirac equation in particle physics.

The commutator of two operators acting on a Hilbert space is a central concept in quantum mechanics, since it quantifies how well the two observables described by these operators can be measured simultaneously. The uncertainty principle is ultimately a theorem about such commutators, by virtue of the Robertson–Schrödinger relation.[7] In phase space, equivalent commutators of function star-products are called Moyal brackets, and are completely isomorphic to the Hilbert-space commutator structures mentioned.

4.2.1 Identities (ring theory)

The commutator has the following properties:

Lie-algebra identities:

1. $[A+B,C] = [A,C] + [B,C]$
2. $[A,A] = 0$
3. $[A,B] = -[B,A]$
4. $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

The third relation is called anticommutativity, while the fourth is the Jacobi identity.

Additional identities:

1. $[A,BC] = [A,B]C + B[A,C]$
2. $[A,BCD] = [A,B]CD + B[A,C]D + BC[A,D]$

3. $[A, BCDE] = [A, B]CDE + B[A, C]DE + BC[A, D]E + BCD[A, E]$
4. $[AB, C] = A[B, C] + [A, C]B$
5. $[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC$
6. $[ABCD, E] = ABC[D, E] + AB[C, E]D + A[B, E]CD + [A, E]BCD$
7. $[AB, CD] = A[B, CD] + [A, CD]B = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$
8. $[[[A, B], C], D] + [[[B, C], D], A] + [[[C, D], A], B] + [[[D, A], B], C] = [[A, C], [B, D]]$

If A is a fixed element of a ring *R*, the first additional identity can be interpreted as a Leibniz rule for the map $\mathrm{ad}_A : R \to R$ given by $\mathrm{ad}_A(B) = [A, B]$. In other words, the map *adA* defines a derivation on the ring *R*. The second and third identities represent Leibniz rules for more than two factors that are valid for any derivation. Identities 4-6 can also be interpreted as Leibniz rules for a certain derivation.

The following useful identity ("Hadamard Lemma") involves nested commutators and underlies the Campbell–Baker–Hausdorff expansion of log (exp *A* exp *B*):

- $e^A Be^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \equiv e^{\mathrm{ad}(A)}B.$

This formula is valid in any ring or algebra where the exponential function can be meaningfully defined, for instance in a Banach algebra or in a ring of formal power series.

Use of the same expansion expresses the above Lie group commutator in terms of a series of nested Lie bracket (algebra) commutators,

- $\ln\left(e^A e^B e^{-A} e^{-B}\right) = [A, B] + \frac{1}{2!}[(A+B), [A, B]] + \frac{1}{3!}\left([A, [B, [B, A]]]/2 + [(A + B), [(A + B), [A, B]]]\right) + \cdots.$

These identities differ slightly for the anticommutator (defined above), for instance

- $\{A, BC\} = \{A, B\}C - B[A, C]$
- $[AB, C] = A\{B, C\} - \{A, C\}B$

4.3 Graded rings and algebras

When dealing with graded algebras, the commutator is usually replaced by the **graded commutator**, defined in homogeneous components as

$$[\omega, \eta]_{gr} := \omega\eta - (-1)^{\deg \omega \deg \eta}\eta\omega.$$

4.4 Derivations

Especially if one deals with multiple commutators, another notation turns out to be useful involving the adjoint representation:

$$\mathrm{ad}(x)(y) = [x, y].$$

Then **ad**(*x*) is a derivation and **ad** is linear,

$$\mathrm{ad}(x + y) = \mathrm{ad}(x) + \mathrm{ad}(y) \text{ and } \mathrm{ad}(\lambda x) = \lambda\ \mathrm{ad}(x)\ ,$$

and, crucially, a Lie algebra homomorphism,

$\operatorname{ad}([x,y]) = [\operatorname{ad}(x), \operatorname{ad}(y)]$.

By contrast, it is **not** always an algebra homomorphism, *i.e.*, a relation $\operatorname{ad}(xy) = \operatorname{ad}(x)\operatorname{ad}(y)$ **does not hold in general**.

Examples

- $\operatorname{ad}(x)\operatorname{ad}(x)(y) = [x, [x, y]]$
- $\operatorname{ad}(x)\operatorname{ad}(a+b)(y) = [x, [a+b, y]]$

4.5 See also

- Anticommutativity
- Associator
- Canonical commutation relation
- Centralizer a.k.a. commutant
- Derivation (abstract algebra)
- Moyal bracket
- Pincherle derivative
- Poisson bracket
- Ternary commutator
- Three subgroups lemma
- Baker–Campbell–Hausdorff formula

4.6 Notes

[1] Fraleigh (1976, p. 108)

[2] Herstein (1964, p. 55)

[3] McKay (2000, p. 4)

[4] Herstein (1964, p. 70)

[5] Fraleigh (1976, p. 128)

[6] McMahon (2008)

[7] Liboff (2003, pp. 140–142)

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4.8 External links

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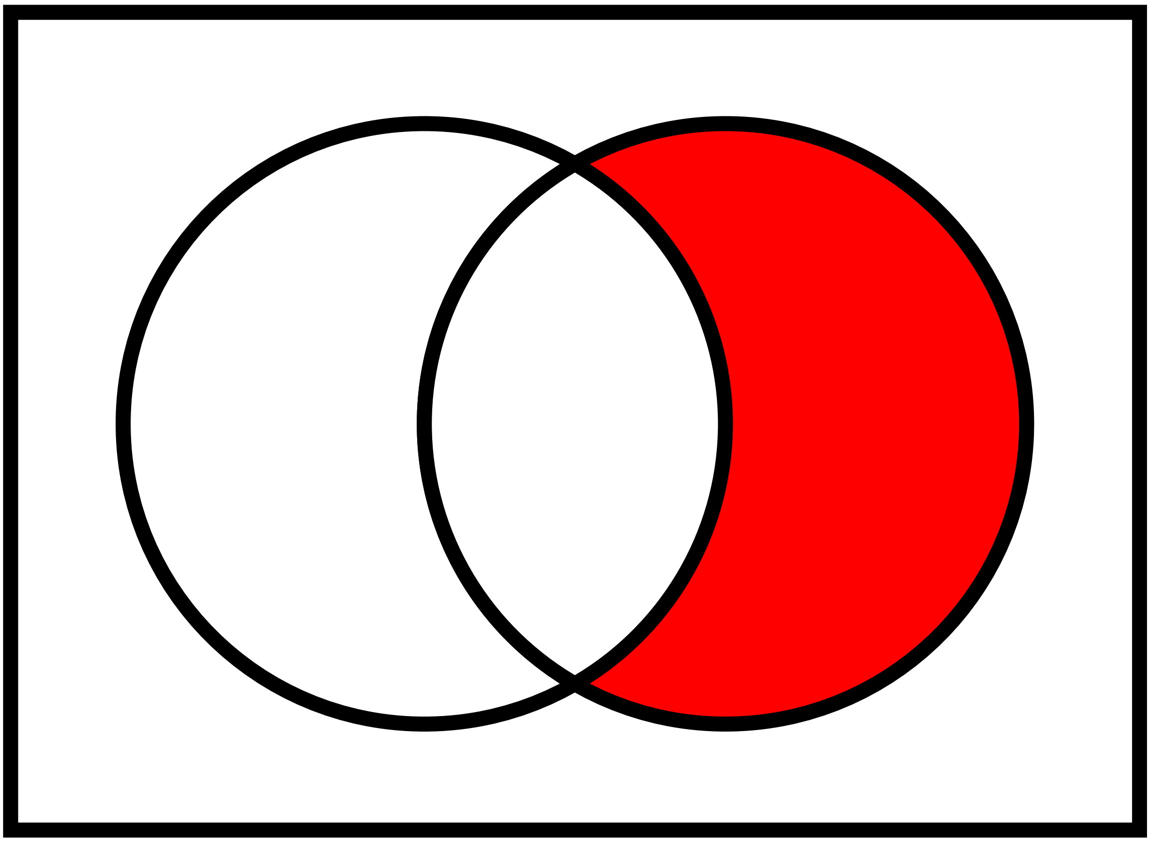
Chapter 5

Complement (set theory)

In set theory, a **complement** of a set A refers to things not in (that is, things outside of) A. The **relative complement** of A with respect to a set B is the set of elements in B but not in A. When all sets under consideration are considered to be subsets of a given set U, the **absolute complement** of A is the set of all elements in U but not in A.

5.1 Relative complement

If A and B are sets, then the **relative complement** of A in B,[1] also termed the **set-theoretic difference** of B and A,[2] is the set of elements in B, but not in A.



*The **relative complement** of* A *(left circle) in* B *(right circle):* $B \cap A^c \quad = \quad B \setminus A$

The relative complement of A in B is denoted $B \setminus A$ according to the ISO 31-11 standard (sometimes written $B - A$, but this notation is ambiguous, as in some contexts it can be interpreted as the set of all $b - a$, where b is taken from B and a from A).

Formally

$B \setminus A = \{x \in B \mid x \notin A\}.$

Examples:

- {1,2,3} \ {2,3,4} = {1}
- {2,3,4} \ {1,2,3} = {4}
- If $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{R} \setminus \mathbb{Q} = \mathbb{I}$ is the set of irrational numbers.

The following lists some notable properties of relative complements in relation to the set-theoretic operations of union and intersection.

If A, B, and C are sets, then the following identities hold:

- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
- $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$
- $C \setminus (B \setminus A) = (C \cap A) \cup (C \setminus B)$,

  with the important special case $C \setminus (C \setminus A) = C \cap A$ demonstrating that intersection can be expressed using only the relative complement operation.

- $(B \setminus A) \cap C = (B \cap C) \setminus A = B \cap (C \setminus A)$
- $(B \setminus A) \cup C = (B \cup C) \setminus (A \setminus C)$
- $A \setminus A = \varnothing$
- $\varnothing \setminus A = \varnothing$
- $A \setminus \varnothing = A$

5.2 Absolute complement

If a universe **U** is defined, then the relative complement of A in **U** is called the **absolute complement** (or simply **complement**) of A, and is denoted by A^c or sometimes A'. The same set often[3] is denoted by $\complement_U A$ or $\complement A$ if **U** is fixed, that is:

$A^c = \mathbf{U} \setminus A.$

For example, if the universe is the set of integers, then the complement of the set of odd numbers is the set of even numbers.

The following lists some important properties of absolute complements in relation to the set-theoretic operations of union and intersection.

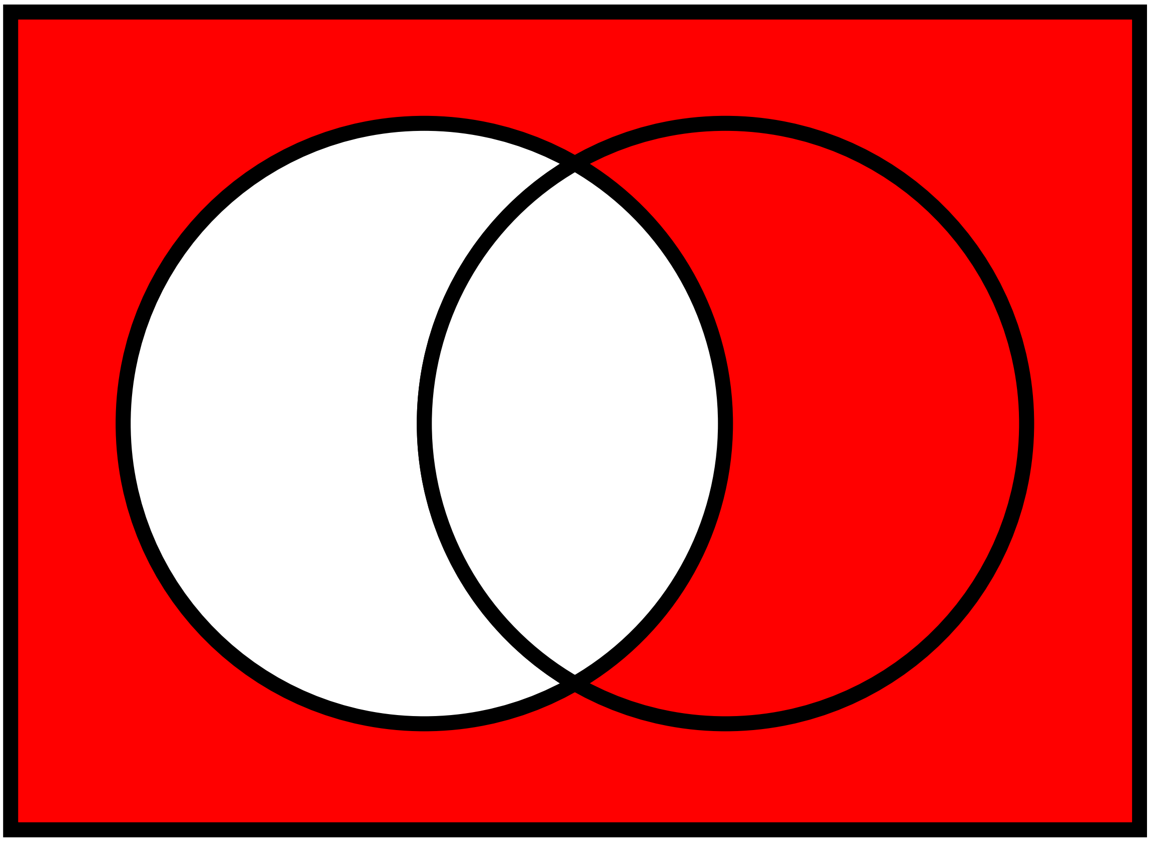
If A and B are subsets of a universe **U**, then the following identities hold:

De Morgan's laws:[1]

- $(A \cup B)^c = A^c \cap B^c.$
- $(A \cap B)^c = A^c \cup B^c.$

Complement laws:[1]

- $A \cup A^c = U.$
- $A \cap A^c = \emptyset.$
- $\emptyset^c = U.$



*The **absolute complement** of A in **U**:* A^c = ***U*** \ A

- $U^c = \emptyset$.
- If $A \subset B$ then ,$B^c \subset A^c$.
  (this follows from the equivalence of a conditional with its contrapositive)

Involution or double complement law:

- $(A^c)^c = A$.

Relationships between relative and absolute complements:

- $A \setminus B = A \cap B^c$.
- $(A \setminus B)^c = A^c \cup B$.

Relationship with set difference:

- $A^c \setminus B^c = B \setminus A$.

The first two complement laws above shows that if A is a non-empty, proper subset of **U**, then $\{A, A^c\}$ is a partition of **U**.

5.3 Notation

In the LaTeX typesetting language, the command \setminus[4] is usually used for rendering a set difference symbol, which is similar to a backslash symbol. When rendered the \setminus command looks identical to \backslash except that it has a little more space in front and behind the slash, akin to the LaTeX sequence \mathbin{\backslash}. A variant \smallsetminus is available in the amssymb package.

5.4 Complements in various programming languages

Some programming languages allow for manipulation of sets as data structures, using these operators or functions to construct the difference of sets a and b:

.NET Framework a.Except(b);

C++ set_difference(a.begin(), a.end(), b.begin(), b.end(), result.begin());

Clojure (clojure.set/difference a b)[5]

Common Lisp set-difference, nset-difference[6]

F# Set.difference a b[7]

or

a - b[8]

Falcon diff = a - b[9]

Haskell difference a b

a \\ b[10]

Java diff = a.clone();

diff.removeAll(b);[11]

Julia setdiff[12]

Mathematica Complement[13]

MATLAB setdiff[14]

OCaml Set.S.diff[15]

Octave setdiff[16]

PARI/GP setminus[17]

Pascal SetDifference := a - b;

Perl 5 #for perl version >= 5.10

@a = grep {not $_ ~~ @b} @a;

Perl 6 $A \ $B

$A (-) $B # texas version

PHP array_diff($a, $b);[18]

Prolog a(X),\+ b(X).

Python diff = a.difference(b)[19]

diff = a - b[19]

R setdiff[20]

Racket (set-subtract a b)[21]

Ruby diff = a - b[22]

Scala a.diff(b)[23]

or

a -- b[23]

Smalltalk (Pharo) a difference: b

SQL SELECT * FROM A

EXCEPT SELECT * FROM B

Unix shell comm −23 a b[24]

grep -vf b a # less efficient, but works with small unsorted sets

5.5 See also

- Algebra of sets
- Naive set theory
- Symmetric difference

5.6 References

[1] Halmos (1960) p.17

[2] Devlin (1979) p.6

[3] Bourbaki p. E II.6

[4] The Comprehensive LaTeX Symbol List

[5] clojure.set API reference

[6] Common Lisp HyperSpec, Function set-difference, nset-difference. Accessed on September 8, 2009.

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[22] Class: Array Ruby Documentation

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5.7 External links

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Chapter 6

Concatenation (mathematics)

In mathematics, **concatenation** is the joining of two numbers by their numerals. That is, the concatenation of 123 and 456 is 123456. Concatenation of numbers *a* and *b* is denoted *a*||*b*. Relevant subjects in recreational mathematics include Smarandache-Wellin numbers, home primes, and Champernowne's constant. The convention for sequences at places such as the Online Encyclopedia of Integer Sequences is to have sequences of concatenations include as the first term a number prior to the actual act of concatenation. Therefore, care must be taken to ensure that parties discussing a topic agree either with this convention or with plain language. For example, the first term in the sequence of concatenations of increasing even numbers may be taken to be either 24, as would seem obviously correct, or simply 2, according to convention.

6.1 Calculation

The concatenation of numbers depends on the numeric base, which is often understood from context.

Given the numbers *p* and *q* in base *b*, the concatenation *p*||*q* is given by

$$p||q = pb^{l(q)} + q$$

where

$$l(q) = \lfloor log_b(q) \rfloor + 1$$

is the number of digits of *q* in base *b*, and $\lfloor x \rfloor$ is the floor function.

6.2 Vector extension

The concatenation of vectors can be understood in two distinct ways; either as a generalization of the above operation for numbers or as a concatenation of lists.

Given two vectors in $\mathbb{R}^n$, concatenation can be defined as

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} || \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1||b_1 \\ a_2||b_2 \\ \vdots \\ a_n||b_n \end{pmatrix}$$

In the case of vectors in $\mathbb{R}^1$, this is equivalent to the above definition for numbers. The further extension to matrices is trivial.

Since vectors can be viewed in a certain way as lists, concatenation may take on another meaning. In this case the concatenation of two lists (a_1, a_2, ..., an) and (b_1, b_2, ..., bn) is the list (a_1, a_2, ..., an, b_1, b_2, ..., bn). Only the exact context will reveal which meaning is intended.

Vector concatenation is commonly done in computationally computing.[1]

6.3 See also

- Concatenation
- Concatenated code

6.4 References

[1] "Creating and Concatenating Matrices". *www.mathworks.com*. Retrieved 20 January 2016.

"MathWorld: Concatenation".

Chapter 7

Connected sum

In mathematics, specifically in topology, the operation of **connected sum** is a geometric modification on manifolds. Its effect is to join two given manifolds together near a chosen point on each. This construction plays a key role in the classification of closed surfaces.

More generally, one can also join manifolds together along identical submanifolds; this generalization is often called the **fiber sum**. There is also a closely related notion of a connected sum on knots, called the **knot sum** or **composition** of knots.

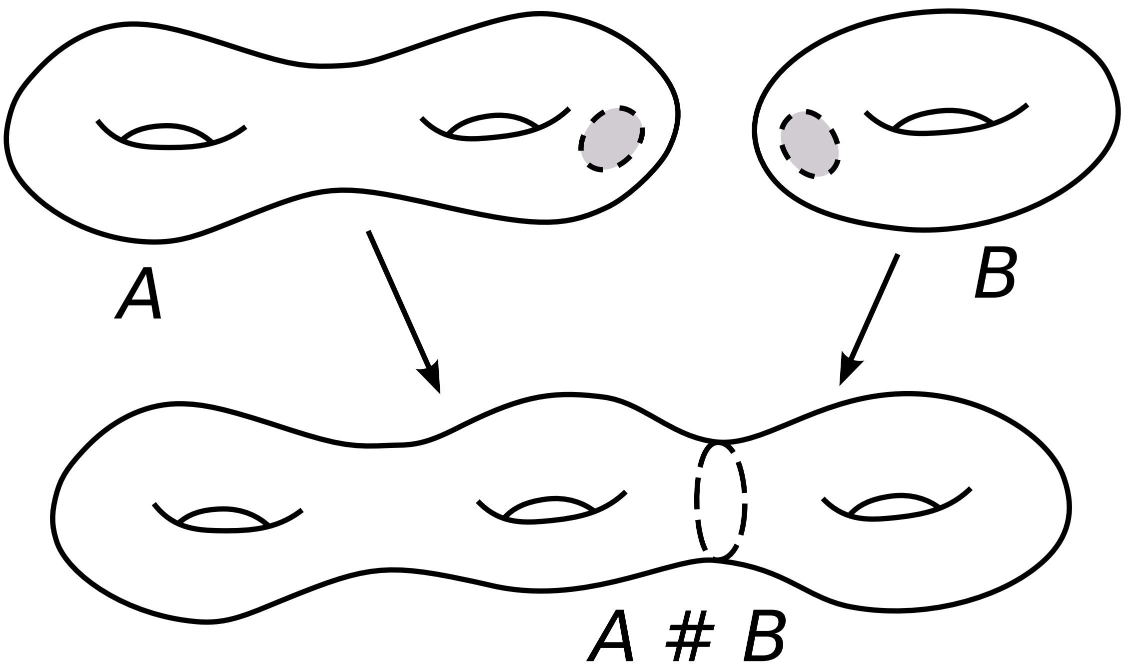


Illustration of connected sum.

7.1 Connected sum at a point

A **connected sum** of two m-dimensional manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

If both manifolds are oriented, there is a unique connected sum defined by having the gluing map reverse orientation. Although the construction uses the choice of the balls, the result is unique up to homeomorphism. One can also make this operation work in the smooth category, and then the result is unique up to diffeomorphism. There are subtle problems in the smooth case: not every diffeomorphism between the boundaries of the spheres gives the same composite manifold, even if the orientations are chosen correctly. For example, Milnor showed that two 7-cells can be glued along their boundary so that the result is an exotic sphere homeomorphic but not diffeomorphic to a 7-sphere. However there is a canonical way to choose the gluing which gives a unique well defined connected sum. This uniqueness depends crucially on the disc theorem, which is not at all obvious.

The operation of connected sum is denoted by # ; for example $A\#B$ denotes the connected sum of A and B .

The operation of connected sum has the sphere S^m as an identity; that is, $M\#S^m$ is homeomorphic (or diffeomorphic) to M .

The classification of closed surfaces, a foundational and historically significant result in topology, states that any closed surface can be expressed as the connected sum of a sphere with some number g of tori and some number k of real projective planes.

7.2 Connected sum along a submanifold

Let M_1 and M_2 be two smooth, oriented manifolds of equal dimension and V a smooth, closed, oriented manifold, embedded as a submanifold into both M_1 and M_2 . Suppose furthermore that there exists an isomorphism of normal bundles

$$\psi : N_{M_1}V \to N_{M_2}V$$

that reverses the orientation on each fiber. Then ψ induces an orientation-preserving diffeomorphism

$$N_1 \setminus V \cong N_{M_1}V \setminus V \to N_{M_2}V \setminus V \cong N_2 \setminus V,$$

where each normal bundle $N_{M_i}V$ is diffeomorphically identified with a neighborhood N_i of V in M_i , and the map

$$N_{M_2}V \setminus V \to N_{M_2}V \setminus V$$

is the orientation-reversing diffeomorphic involution

$$v \mapsto v/|v|^2$$

on normal vectors. The **connected sum** of M_1 and M_2 along V is then the space

$$(M_1 \setminus V) \bigcup_{N_1 \setminus V = N_2 \setminus V} (M_2 \setminus V)$$

obtained by gluing the deleted neighborhoods together by the orientation-preserving diffeomorphism. The sum is often denoted

$$(M_1, V)\#(M_2, V).$$

Its diffeomorphism type depends on the choice of the two embeddings of V and on the choice of ψ .

Loosely speaking, each normal fiber of the submanifold V contains a single point of V , and the connected sum along V is simply the connected sum as described in the preceding section, performed along each fiber. For this reason, the connected sum along V is often called the **fiber sum**.

The special case of V a point recovers the connected sum of the preceding section.

7.3 Connected sum along a codimension-two submanifold

Another important special case occurs when the dimension of V is two less than that of the M_i . Then the isomorphism ψ of normal bundles exists whenever their Euler classes are opposite:

$$e(N_{M_1}V) = -e(N_{M_2}V).$$

Furthermore, in this case the structure group of the normal bundles is the circle group $SO(2)$; it follows that the choice of embeddings can be canonically identified with the group of homotopy classes of maps from V to the circle, which in turn equals the first integral cohomology group $H^1(V)$. So the diffeomorphism type of the sum depends on the choice of ψ and a choice of element from $H^1(V)$.

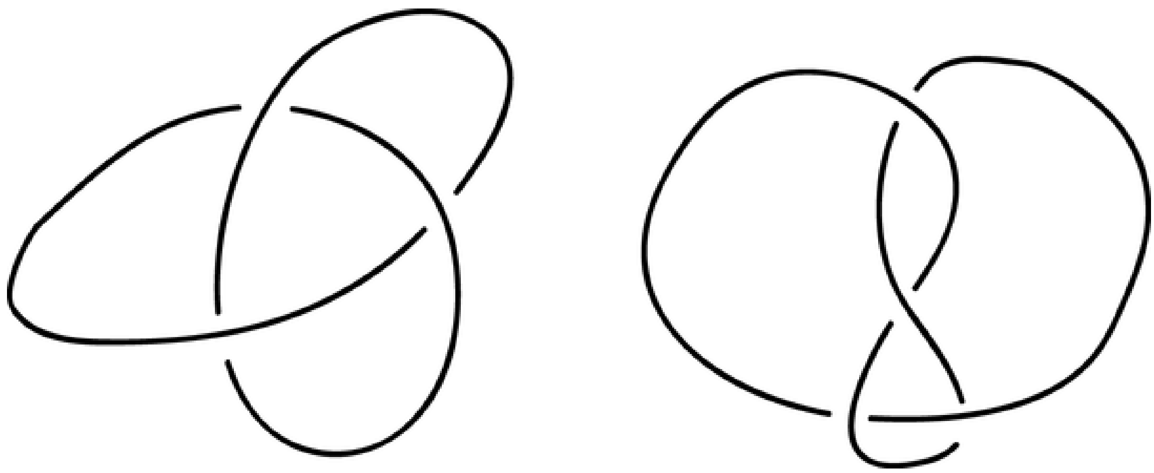
A connected sum along a codimension-two V can also be carried out in the category of symplectic manifolds; this elaboration is called the symplectic sum.

7.4 Local operation

The connected sum is a local operation on manifolds, meaning that it alters the summands only in a neighborhood of V . This implies, for example, that the sum can be carried out on a single manifold M containing two disjoint copies of V , with the effect of gluing M to itself. For example, the connected sum of a two-sphere at two distinct points of the sphere produces the two-torus.

7.5 Connected sum of knots

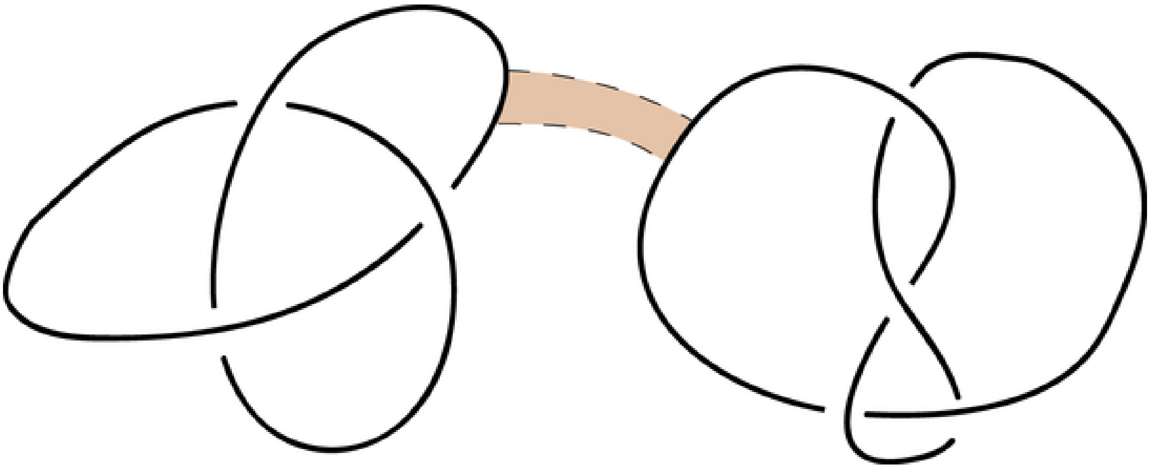
There is a closely related notion of the connected sum of two knots. In fact, if one regards a knot merely as a one-manifold, then the connected sum of two knots is just their connected sum as a one-dimensional manifold. However, the essential property of a knot is not its manifold structure (under which every knot is equivalent to a circle) but rather its embedding into the ambient space. So the connected sum of knots has a more elaborate definition that produces a well-defined embedding, as follows.



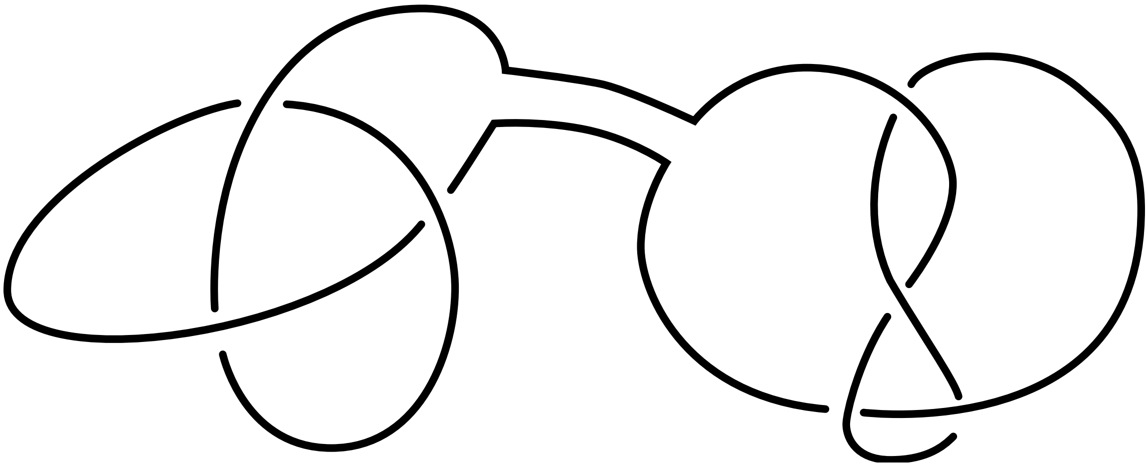
Consider disjoint planar projections of each knot.

This procedure results in the projection of a new knot, a **connected sum** (or **knot sum**, or **composition**) of the original knots. For the connected sum of knots to be well defined, one has to consider **oriented knots** in 3-space. To define the connected sum for two oriented knots:

1. Consider a planar projection of each knot and suppose these projections are disjoint.
2. Find a rectangle in the plane where one pair of sides are arcs along each knot but is otherwise disjoint from the knots **and** so that the arcs of the knots on the sides of the rectangle are oriented around the boundary of the rectangle in the **same direction**.



Find a rectangle in the plane where one pair of sides are arcs along each knot but is otherwise disjoint from the knots.



Now join the two knots together by deleting these arcs from the knots and adding the arcs that form the other pair of sides of the rectangle.

3. Now join the two knots together by deleting these arcs from the knots and adding the arcs that form the other pair of sides of the rectangle.

The resulting connected sum knot inherits an orientation consistent with the orientations of the two original knots, and the oriented ambient isotopy class of the result is well-defined, depending only on the oriented ambient isotopy classes of the original two knots.

Under this operation, oriented knots in 3-space form a commutative monoid with unique prime factorization, which allows us to define what is meant by a prime knot. Proof of commutativity can be seen by letting one summand shrink until it is very small and then pulling it along the other knot. The unknot is the unit. The two trefoil knots are the simplest prime knots. Higher-dimensional knots can be added by splicing the n -spheres.

In three dimensions, the unknot cannot be written as the sum of two non-trivial knots. This fact follows from additivity of knot genus; another proof relies on an infinite construction sometimes called the Mazur swindle. In higher dimensions (with codimension at least three), it is possible to get an unknot by adding two nontrivial knots.

If one does **not** take into account the orientations of the knots, the connected sum operation is not well defined on isotopy classes of (nonoriented) knots. To see this, consider two noninvertible knots K, L which are not equivalent (as unoriented knots); for example take the two pretzel knots $K = P(3,5,7)$ and $L = P(3,5,9)$. Let K_+ and K- be K with its two inequivalent orientations, and let L_+ and L- be L with its two inequivalent orientations. There are four oriented connected sums we may form:

- $A = K_+ \# L_+$

- $B = K\text{-} \# L\text{-}$
- $C = K_{+} \# L\text{-}$
- $D = K\text{-} \# L_{+}$

The oriented ambient isotopy classes of these four oriented knots are all distinct. And, when one considers ambient isotopy of the knots without regard to orientation, there are **two distinct** equivalence classes: { $A \sim B$ } and { $C \sim D$ }. To see that *A* and *B* are unoriented equivalent, simply note that they both may be constructed from the same pair of disjoint knot projections as above, the only difference being the orientations of the knots. Similarly, one sees that *C* and *D* may be constructed from the same pair of disjoint knot projections.

7.6 See also

- Band sum
- Prime decomposition (3-manifold)
- Manifold decomposition
- Satellite knot

7.7 Further reading

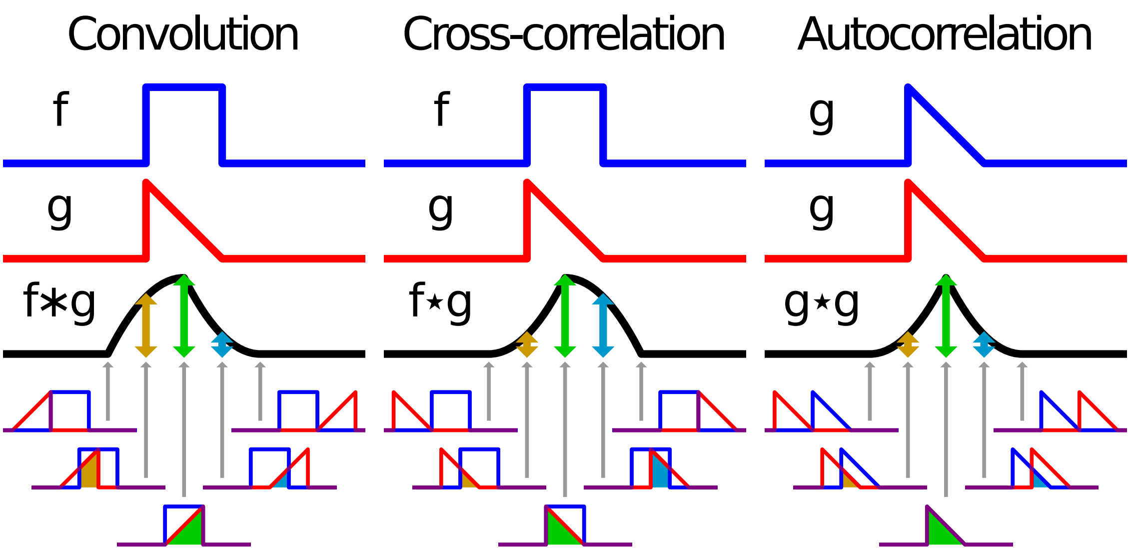
- Robert Gompf: A new construction of symplectic manifolds, *Annals of Mathematics* 142 (1995), 527–595
- William S. Massey, *A Basic Course in Algebraic Topology*, Springer-Verlag, 1991. ISBN 0-387-97430-X.

Chapter 8

Convolution

For the usage in formal language theory, see Convolution (computer science). For other uses, see Convolute.

In mathematics and, in particular, functional analysis, **convolution** is a mathematical operation on two functions



Visual comparison of convolution, cross-correlation and autocorrelation.

f and g, producing a third function that is typically viewed as a modified version of one of the original functions, giving the integral of the pointwise multiplication of the two functions as a function of the amount that one of the original functions is translated. Convolution is similar to cross-correlation. It has applications that include probability, statistics, computer vision, natural language processing, image and signal processing, engineering, and differential equations.

The convolution can be defined for functions on groups other than Euclidean space. For example, periodic functions, such as the discrete-time Fourier transform, can be defined on a circle and convolved by *periodic convolution*. (See row 10 at DTFT#Properties.) A *discrete convolution* can be defined for functions on the set of integers. Generalizations of convolution have applications in the field of numerical analysis and numerical linear algebra, and in the design and implementation of finite impulse response filters in signal processing.

Computing the inverse of the convolution operation is known as deconvolution.

8.1 Definition

The convolution of f and g is written $f*g$, using an asterisk or star. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

While the symbol *t* is used above, it need not represent the time domain. But in that context, the convolution formula can be described as a weighted average of the function $f(\tau)$ at the moment *t* where the weighting is given by $g(-\tau)$ simply shifted by amount *t*. As *t* changes, the weighting function emphasizes different parts of the input function.

For functions *f*, *g* supported on only $[0, \infty)$ (i.e., zero for negative arguments), the integration limits can be truncated, resulting in

In this case, the Laplace transform is more appropriate than the Fourier transform below and boundary terms become relevant.

For the multi-dimensional formulation of convolution, see Domain of definition (below).

8.1.1 Derivations

Convolution describes the output (in terms of the input) of an important class of operations known as *linear time-invariant* (LTI). See LTI system theory for a derivation of convolution as the result of LTI constraints. In terms of the Fourier transforms of the input and output of an LTI operation, no new frequency components are created. The existing ones are only modified (amplitude and/or phase). In other words, the output transform is the pointwise product of the input transform with a third transform (known as a transfer function). See Convolution theorem for a derivation of that property of convolution. Conversely, convolution can be derived as the inverse Fourier transform of the pointwise product of two Fourier transforms.

8.2 Visual explanation

8.3 Historical developments

One of the earliest uses of the convolution integral appeared in D'Alembert's derivation of Taylor's theorem in *Recherches sur différents points importants du système du monde,* published in 1754.[1]

Also, an expression of the type:

$$\int f(u) \cdot g(x-u)du$$

is used by Sylvestre François Lacroix on page 505 of his book entitled *Treatise on differences and series*, which is the last of 3 volumes of the encyclopedic series: *Traité du calcul différentiel et du calcul intégral*, Chez Courcier, Paris, 1797-1800.[2] Soon thereafter, convolution operations appear in the works of Pierre Simon Laplace, Jean Baptiste Joseph Fourier, Siméon Denis Poisson, and others. The term itself did not come into wide use until the 1950s or 60s. Prior to that it was sometimes known as *faltung* (which means *folding* in German), *composition product*, *superposition integral*, and *Carson's integral*.[3] Yet it appears as early as 1903, though the definition is rather unfamiliar in older uses.[4][5]

The operation:

$$\int_0^t \varphi(s)\psi(t-s)\,ds, \qquad 0 \le t < \infty,$$

is a particular case of composition products considered by the Italian mathematician Vito Volterra in 1913.[6]

8.4 Circular convolution

Main article: Circular convolution

When a function g_T is periodic, with period T, then for functions, f, such that $f*g_T$ exists, the convolution is also periodic and identical to:

$$(f * g_T)(t) \equiv \int_{t_0}^{t_0+T} \left[\sum_{k=-\infty}^{\infty} f(\tau + kT)\right] g_T(t-\tau)\, d\tau,$$

where t_0 is an arbitrary choice. The summation is called a periodic summation of the function f.

When g_T is a periodic summation of another function, g, then $f*g_T$ is known as a *circular* or *cyclic* convolution of f and g.
And if the periodic summation above is replaced by f_T, the operation is called a *periodic* convolution of f_T and g_T.

8.5 Discrete convolution

For complex-valued functions f, g defined on the set **Z** of integers, the **discrete convolution** of f and g is given by:[7]

$$\begin{aligned}(f * g)[n] &\stackrel{\text{def}}{=} \textstyle\sum_{m=-\infty}^{\infty} f[m]\, g[n-m] \\ &= \textstyle\sum_{m=-\infty}^{\infty} f[n-m]\, g[m]. \text{ (commutativity)}\end{aligned}$$

The convolution of two finite sequences is defined by extending the sequences to finitely supported functions on the set of integers. When the sequences are the coefficients of two polynomials, then the coefficients of the ordinary product of the two polynomials are the convolution of the original two sequences. This is known as the Cauchy product of the coefficients of the sequences.

Thus when g has finite support in the set $\{-M, -M+1, \dots, M-1, M\}$ (representing, for instance, a finite impulse response), a finite summation may be used:[8]

$$(f * g)[n] = \sum_{m=-M}^{M} f[n-m] g[m].$$

8.6 Circular discrete convolution

When a function g_N is periodic, with period N, then for functions, f, such that $f*g_N$ exists, the convolution is also periodic and identical to:

$$(f * g_N)[n] \equiv \sum_{m=0}^{N-1} \left(\sum_{k=-\infty}^{\infty} f[m + kN]\right) g_N[n-m].$$

The summation on k is called a periodic summation of the function f.

If g_N is a periodic summation of another function, g, then $f*g_N$ is known as a circular convolution of f and g.

When the non-zero durations of both f and g are limited to the interval $[0, N-1]$, $f*g_N$ reduces to these common forms:

The notation ($f *_N g$) for *cyclic convolution* denotes convolution over the cyclic group of integers modulo N.

Circular convolution arises most often in the context of fast convolution with an FFT algorithm.

8.6.1 Fast convolution algorithms

In many situations, discrete convolutions can be converted to circular convolutions so that fast transforms with a convolution property can be used to implement the computation. For example, convolution of digit sequences is the kernel operation in multiplication of multi-digit numbers, which can therefore be efficiently implemented with transform techniques (Knuth 1997, §4.3.3.C; von zur Gathen & Gerhard 2003, §8.2).

Eq.1 requires N arithmetic operations per output value and N^2 operations for N outputs. That can be significantly reduced with any of several fast algorithms. Digital signal processing and other applications typically use fast convolution algorithms to reduce the cost of the convolution to O(N log N) complexity.

The most common fast convolution algorithms use fast Fourier transform (FFT) algorithms via the circular convolution theorem. Specifically, the circular convolution of two finite-length sequences is found by taking an FFT of each sequence, multiplying pointwise, and then performing an inverse FFT. Convolutions of the type defined above are then efficiently implemented using that technique in conjunction with zero-extension and/or discarding portions of the output. Other fast convolution algorithms, such as the Schönhage–Strassen algorithm or the Mersenne transform,[9] use fast Fourier transforms in other rings.

If one sequence is much longer than the other, zero-extension of the shorter sequence and fast circular convolution is not the most computationally efficient method available.[10] Instead, decomposing the longer sequence into blocks and convolving each block allows for faster algorithms such as the Overlap–save method and Overlap–add method.[11] A hybrid convolution method that combines block and FIR algorithms allows for a zero input-output latency that is useful for real-time convolution computations.[12]

8.7 Domain of definition

The convolution of two complex-valued functions on $\mathbf{R}^d$ is itself a complex-valued function on $\mathbf{R}^d$, defined by:

$$(f * g)(x) = \int_{\mathbf{R}^d} f(y)g(x-y)\,dy = \int_{\mathbf{R}^d} f(x-y)g(y)\,dy,$$

is well-defined only if f and g decay sufficiently rapidly at infinity in order for the integral to exist. Conditions for the existence of the convolution may be tricky, since a blow-up in g at infinity can be easily offset by sufficiently rapid decay in f. The question of existence thus may involve different conditions on f and g:

8.7.1 Compactly supported functions

If f and g are compactly supported continuous functions, then their convolution exists, and is also compactly supported and continuous (Hörmander 1983, Chapter 1). More generally, if either function (say f) is compactly supported and the other is locally integrable, then the convolution $f*g$ is well-defined and continuous.

Convolution of f and g is also well defined when both functions are locally square integrable on $\mathbf{R}$ and supported on an interval of the form [a, +∞) (or both supported on [-∞, a]).

8.7.2 Integrable functions

The convolution of f and g exists if f and g are both Lebesgue integrable functions in $L^1(\mathbf{R}^d)$, and in this case $f*g$ is also integrable (Stein & Weiss 1971, Theorem 1.3). This is a consequence of Tonelli's theorem. This is also true for functions in ℓ^1 , under the discrete convolution, or more generally for the convolution on any group.

Likewise, if $f \in L^1(\mathbf{R}^d)$ and $g \in L^p(\mathbf{R}^d)$ where $1 \le p \le \infty$, then $f*g \in L^p(\mathbf{R}^d)$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

In the particular case $p = 1$, this shows that L^1 is a Banach algebra under the convolution (and equality of the two sides holds if f and g are non-negative almost everywhere).

More generally, Young's inequality implies that the convolution is a continuous bilinear map between suitable L^p spaces. Specifically, if $1 \le p,q,r \le \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

then

$$\|f * g\|_r \le \|f\|_p \|g\|_q, \quad f \in \mathcal{L}^p,\ g \in \mathcal{L}^q,$$

so that the convolution is a continuous bilinear mapping from $L^p \times L^q$ to L^r. The Young inequality for convolution is also true in other contexts (circle group, convolution on **Z**). The preceding inequality is not sharp on the real line: when $1 < p, q, r < \infty$, there exists a constant $Bp, q < 1$ such that:

$$\|f * g\|_r \le B_{p,q} \|f\|_p \|g\|_q, \quad f \in \mathcal{L}^p,\ g \in \mathcal{L}^q.$$

The optimal value of Bp, q was discovered in 1975.[13]

A stronger estimate is true provided $1 < p, q, r < \infty$:

$$\|f * g\|_r \le C_{p,q} \|f\|_p \|g\|_{q,w}$$

where $\|g\|_{q,w}$ is the weak L^q norm. Convolution also defines a bilinear continuous map $L^{p,w} \times L^{q,w} \to L^{r,w}$ for $1 < p, q, r < \infty$, owing to the weak Young inequality:[14]

$$\|f * g\|_{r,w} \le C_{p,q} \|f\|_{p,w} \|g\|_{r,w}.$$

8.7.3 Functions of rapid decay

In addition to compactly supported functions and integrable functions, functions that have sufficiently rapid decay at infinity can also be convolved. An important feature of the convolution is that if f and g both decay rapidly, then $f*g$ also decays rapidly. In particular, if f and g are rapidly decreasing functions, then so is the convolution $f*g$. Combined with the fact that convolution commutes with differentiation (see **Properties**), it follows that the class of Schwartz functions is closed under convolution (Stein & Weiss 1971, Theorem 3.3).

8.7.4 Distributions

Main article: Distribution (mathematics)

Under some circumstances, it is possible to define the convolution of a function with a distribution, or of two distributions. If f is a compactly supported function and g is a distribution, then $f*g$ is a smooth function defined by a distributional formula analogous to

$$\int_{\mathbf{R}^d} f(y) g(x-y)\, dy.$$

More generally, it is possible to extend the definition of the convolution in a unique way so that the associative law

$$f * (g * \varphi) = (f * g) * \varphi$$

remains valid in the case where f is a distribution, and g a compactly supported distribution (Hörmander 1983, §4.2).

8.7.5 Measures

The convolution of any two Borel measures μ and ν of bounded variation is the measure λ defined by (Rudin 1962)

$$\int_{\mathbf{R}^d} f(x)d\lambda(x) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(x+y)\,d\mu(x)\,d\nu(y).$$

This agrees with the convolution defined above when μ and ν are regarded as distributions, as well as the convolution of L^1 functions when μ and ν are absolutely continuous with respect to the Lebesgue measure.

The convolution of measures also satisfies the following version of Young's inequality

$$\|\mu * \nu\| \le \|\mu\|\|\nu\|$$

where the norm is the total variation of a measure. Because the space of measures of bounded variation is a Banach space, convolution of measures can be treated with standard methods of functional analysis that may not apply for the convolution of distributions.

8.8 Properties

8.8.1 Algebraic properties

See also: Convolution algebra

The convolution defines a product on the linear space of integrable functions. This product satisfies the following algebraic properties, which formally mean that the space of integrable functions with the product given by convolution is a commutative algebra without identity (Strichartz 1994, §3.3). Other linear spaces of functions, such as the space of continuous functions of compact support, are closed under the convolution, and so also form commutative algebras.

Commutativity $f * g = g * f$

Associativity $f * (g * h) = (f * g) * h$

Distributivity $f * (g + h) = (f * g) + (f * h)$

Associativity with scalar multiplication $a(f * g) = (af) * g$

for any real (or complex) number a .

Multiplicative identity

No algebra of functions possesses an identity for the convolution. The lack of identity is typically not a major inconvenience, since most collections of functions on which the convolution is performed can be convolved with a delta distribution or, at the very least (as is the case of L^1) admit approximations to the identity. The linear space of compactly supported distributions does, however, admit an identity under the convolution. Specifically,

$$f * \delta = f$$

where δ is the delta distribution.

Inverse element

Some distributions have an inverse element for the convolution, $S^{(-1)}$, which is defined by

$$S^{(-1)} * S = \delta.$$

The set of invertible distributions forms an abelian group under the convolution.

Complex conjugation

$$\overline{f * g} = \overline{f} * \overline{g}$$

8.8.2 Integration

If f and g are integrable functions, then the integral of their convolution on the whole space is simply obtained as the product of their integrals:

$$\int_{\mathbf{R}^d} (f * g)(x)\, dx = \left(\int_{\mathbf{R}^d} f(x)\, dx\right) \left(\int_{\mathbf{R}^d} g(x)\, dx\right).$$

This follows from Fubini's theorem. The same result holds if f and g are only assumed to be nonnegative measurable functions, by Tonelli's theorem.

8.8.3 Differentiation

In the one-variable case,

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

where d/dx is the derivative. More generally, in the case of functions of several variables, an analogous formula holds with the partial derivative:

$$\frac{\partial}{\partial x_i}(f * g) = \frac{\partial f}{\partial x_i} * g = f * \frac{\partial g}{\partial x_i}.$$

A particular consequence of this is that the convolution can be viewed as a "smoothing" operation: the convolution of f and g is differentiable as many times as f and g are in total.

These identities hold under the precise condition that f and g are absolutely integrable and at least one of them has an absolutely integrable (L^1) weak derivative, as a consequence of Young's inequality. For instance, when f is continuously differentiable with compact support, and g is an arbitrary locally integrable function,

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g.$$

These identities also hold much more broadly in the sense of tempered distributions if one of f or g is a compactly supported distribution or a Schwartz function and the other is a tempered distribution. On the other hand, two positive integrable and infinitely differentiable functions may have a nowhere continuous convolution.

In the discrete case, the difference operator $D f(n) = f(n + 1) - f(n)$ satisfies an analogous relationship:

$$D(f * g) = (Df) * g = f * (Dg).$$

8.8.4 Convolution theorem

The convolution theorem states that

$$\mathcal{F}\{f * g\} = k \cdot \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$$

where $\mathcal{F}\{f\}$ denotes the Fourier transform of f , and k is a constant that depends on the specific normalization of the Fourier transform. Versions of this theorem also hold for the Laplace transform, two-sided Laplace transform, Z-transform and Mellin transform.

See also the less trivial Titchmarsh convolution theorem.

8.8.5 Translation invariance

The convolution commutes with translations, meaning that

$$\tau_x(f * g) = (\tau_x f) * g = f * (\tau_x g)$$

where τxf is the translation of the function *f* by *x* defined by

$$(\tau_x f)(y) = f(y - x).$$

If *f* is a Schwartz function, then τx*f* is the convolution with a translated Dirac delta function τx*f* = *f*∗τx *δ*. So translation invariance of the convolution of Schwartz functions is a consequence of the associativity of convolution.

Furthermore, under certain conditions, convolution is the most general translation invariant operation. Informally speaking, the following holds

- Suppose that *S* is a linear operator acting on functions which commutes with translations: *S*(τx*f*) = τx(*Sf*) for all *x*. Then *S* is given as convolution with a function (or distribution) *gS*; that is *Sf* = *gS*∗*f*.

Thus any translation invariant operation can be represented as a convolution. Convolutions play an important role in the study of time-invariant systems, and especially LTI system theory. The representing function *gS* is the impulse response of the transformation *S*.

A more precise version of the theorem quoted above requires specifying the class of functions on which the convolution is defined, and also requires assuming in addition that *S* must be a continuous linear operator with respect to the appropriate topology. It is known, for instance, that every continuous translation invariant continuous linear operator on L^1 is the convolution with a finite Borel measure. More generally, every continuous translation invariant continuous linear operator on L^p for $1 \le p < \infty$ is the convolution with a tempered distribution whose Fourier transform is bounded. To wit, they are all given by bounded Fourier multipliers.

8.9 Convolutions on groups

If *G* is a suitable group endowed with a measure λ, and if *f* and *g* are real or complex valued integrable functions on *G*, then we can define their convolution by

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, d\lambda(y).$$

It is not commutative in general. In typical cases of interest *G* is a locally compact Hausdorff topological group and λ is a (left-) Haar measure. In that case, unless *G* is unimodular, the convolution defined in this way is not the same as $\int f(xy^{-1}) g(y)\, d\lambda(y)$. The preference of one over the other is made so that convolution with a fixed function *g* commutes with left translation in the group:

$$L_h(f * g) = (L_h f) * g.$$

Furthermore, the convention is also required for consistency with the definition of the convolution of measures given below. However, with a right instead of a left Haar measure, the latter integral is preferred over the former.

On locally compact abelian groups, a version of the convolution theorem holds: the Fourier transform of a convolution is the pointwise product of the Fourier transforms. The circle group **T** with the Lebesgue measure is an immediate example. For a fixed g in $L^1(\mathbf{T})$, we have the following familiar operator acting on the Hilbert space $L^2(\mathbf{T})$:

$$Tf(x) = \frac{1}{2\pi}\int_{\mathbf{T}} f(y)g(x-y)\,dy.$$

The operator T is compact. A direct calculation shows that its adjoint T^* is convolution with

$$\bar{g}(-y).$$

By the commutativity property cited above, T is normal: $T^*T = TT^*$. Also, T commutes with the translation operators. Consider the family S of operators consisting of all such convolutions and the translation operators. Then S is a commuting family of normal operators. According to spectral theory, there exists an orthonormal basis $\{hk\}$ that simultaneously diagonalizes S. This characterizes convolutions on the circle. Specifically, we have

$$h_k(x) = e^{ikx}, \quad k \in \mathbb{Z},$$

which are precisely the characters of **T**. Each convolution is a compact multiplication operator in this basis. This can be viewed as a version of the convolution theorem discussed above.

A discrete example is a finite cyclic group of order n. Convolution operators are here represented by circulant matrices, and can be diagonalized by the discrete Fourier transform.

A similar result holds for compact groups (not necessarily abelian): the matrix coefficients of finite-dimensional unitary representations form an orthonormal basis in L^2 by the Peter–Weyl theorem, and an analog of the convolution theorem continues to hold, along with many other aspects of harmonic analysis that depend on the Fourier transform.

8.10 Convolution of measures

Let G be a topological group. If μ and ν are finite Borel measures on G, then their convolution μ∗ν is defined by

$$(\mu * \nu)(E) = \iint 1_E(xy)\,d\mu(x)\,d\nu(y)$$

for each measurable subset E of G. The convolution is also a finite measure, whose total variation satisfies

$$\|\mu * \nu\| \le \|\mu\|\|\nu\|.$$

In the case when G is locally compact with (left-)Haar measure λ, and μ and ν are absolutely continuous with respect to a λ, so that each has a density function, then the convolution μ∗ν is also absolutely continuous, and its density function is just the convolution of the two separate density functions.

If μ and ν are probability measures on the topological group (**R**,+), then the convolution μ∗ν is the probability distribution of the sum $X + Y$ of two independent random variables X and Y whose respective distributions are μ and ν.

8.11 Bialgebras

Let (X, Δ, ∇, ε, η) be a bialgebra with comultiplication Δ, multiplication ∇, unit η, and counit ε. The convolution is a product defined on the endomorphism algebra End(X) as follows. Let φ, ψ ∈ End(X), that is, φ,ψ : $X \to X$ are functions that respect all algebraic structure of X, then the convolution φ∗ψ is defined as the composition

$$X \xrightarrow{\Delta} X \otimes X \xrightarrow{\phi\otimes\psi} X \otimes X \xrightarrow{\nabla} X.$$

The convolution appears notably in the definition of Hopf algebras (Kassel 1995, §III.3). A bialgebra is a Hopf algebra if and only if it has an antipode: an endomorphism S such that

$$S * \mathrm{id}_X = \mathrm{id}_X * S = \eta \circ \varepsilon.$$

8.12 Applications

Convolution and related operations are found in many applications in science, engineering and mathematics.

- In image processing

See also: digital signal processing

> In digital image processing convolutional filtering plays an important role in many important algorithms in edge detection and related processes.
>
> In optics, an out-of-focus photograph is a convolution of the sharp image with a lens function. The photographic term for this is bokeh.
>
> In image processing applications such as adding blurring.

- In digital data processing

> In analytical chemistry, Savitzky–Golay smoothing filters are used for the analysis of spectroscopic data. They can improve signal-to-noise ratio with minimal distortion of the spectra.
>
> In statistics, a weighted moving average is a convolution.

- In acoustics, reverberation is the convolution of the original sound with echoes from objects surrounding the sound source.

> In digital signal processing, convolution is used to map the impulse response of a real room on a digital audio signal.
>
> In electronic music convolution is the imposition of a spectral or rhythmic structure on a sound. Often this envelope or structure is taken from another sound. The convolution of two signals is the filtering of one through the other.[15]

- In electrical engineering, the convolution of one function (the input signal) with a second function (the impulse response) gives the output of a linear time-invariant system (LTI). At any given moment, the output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence (expressed as a multiplicative factor). The impulse response function provides that factor as a function of the elapsed time since each input value occurred.
- In physics, wherever there is a linear system with a "superposition principle", a convolution operation makes an appearance. For instance, in spectroscopy line broadening due to the Doppler effect on its own gives a Gaussian spectral line shape and collision broadening alone gives a Lorentzian line shape. When both effects are operative, the line shape is a convolution of Gaussian and Lorentzian, a Voigt function.

In Time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses, and the measured fluorescence is a sum of exponential decays from each delta pulse.

In computational fluid dynamics, the large eddy simulation (LES) turbulence model uses the convolution operation to lower the range of length scales necessary in computation thereby reducing computational cost.

- In probability theory, the probability distribution of the sum of two independent random variables is the convolution of their individual distributions.

  In kernel density estimation, a distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian. (Diggle 1995).

- In radiotherapy treatment planning systems, most part of all modern codes of calculation applies a convolution-superposition algorithm.

8.13 See also

- Analog signal processing
- Circulant matrix
- Convolution for optical broad-beam responses in scattering media
- Convolution power
- Cross-correlation
- Deconvolution
- Dirichlet convolution
- Jan Mikusinski
- List of convolutions of probability distributions
- LTI system theory#Impulse response and convolution
- Scaled correlation
- Titchmarsh convolution theorem
- Toeplitz matrix (convolutions can be considered a Toeplitz matrix operation where each row is a shifted copy of the convolution kernel)
- Multidimensional discrete convolution

8.14 Notes

[1] Dominguez-Torres, p 2

[2] Dominguez-Torres, p 4

[3] R. N. Bracewell (2005), "Early work on imaging theory in radio astronomy", in W. T. Sullivan, *The Early Years of Radio Astronomy: Reflections Fifty Years After Jansky's Discovery*, Cambridge University Press, p. 172, ISBN 978-0-521-61602-7

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[6] According to [Lothar von Wolfersdorf (2000), “Einige Klassen quadratischer Integralgleichungen”, *Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig*, *Mathematisch-naturwissenschaftliche Klasse*, volume **128**, number 2, 6–7], the source is Volterra, Vito (1913), “Leçons sur les fonctions de linges”. Gauthier-Villars, Paris 1913.

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[14] Reed & Simon 1975, IX.4

[15] Zölzer, Udo, ed. (2002). *DAFX:Digital Audio Effects*, p.48–49. ISBN 0471490784.

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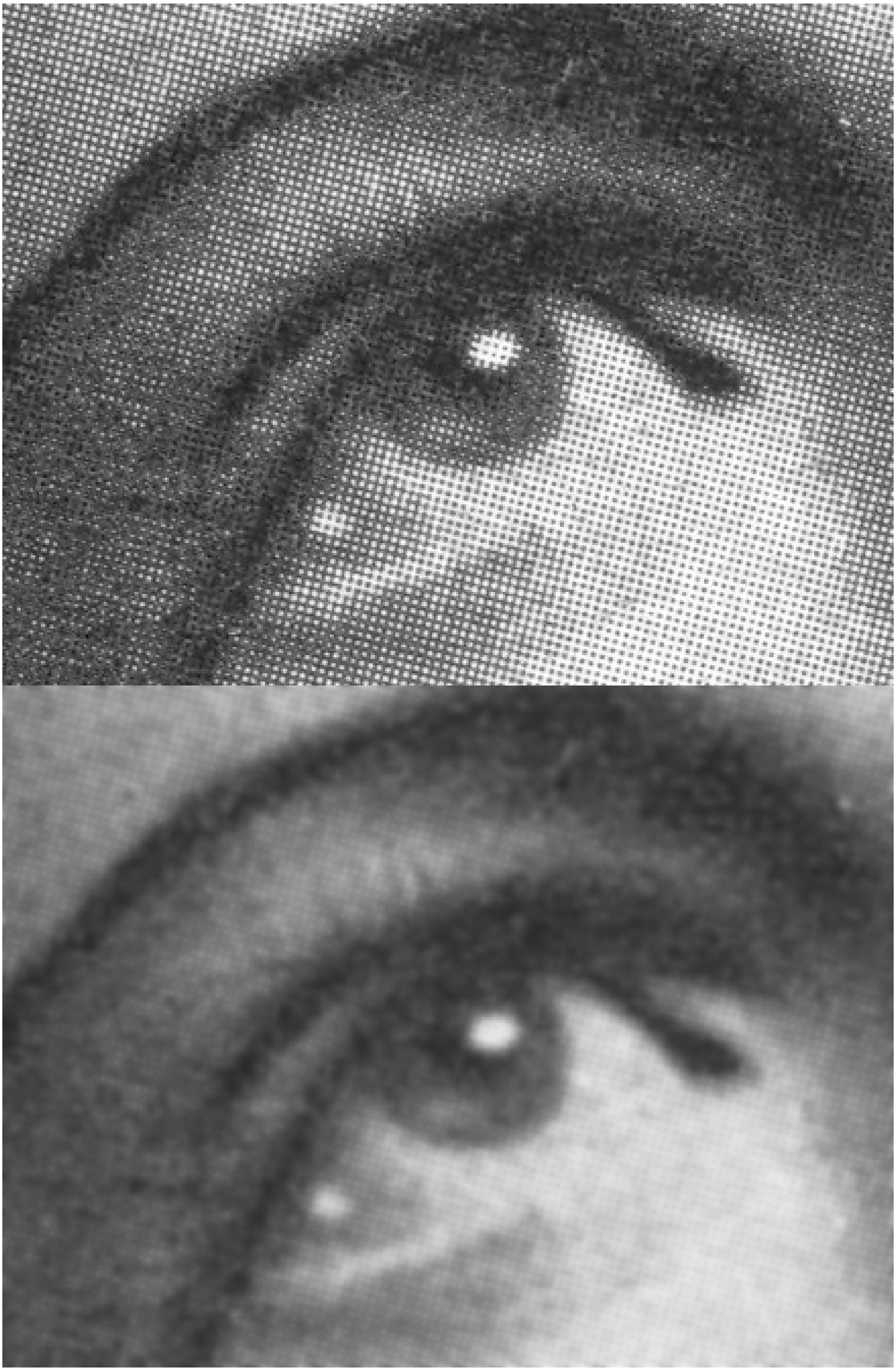
8.16 External links

- Earliest Uses: The entry on Convolution has some historical information.
- Convolution, on The Data Analysis BriefBook
- http://www.jhu.edu/~{}signals/convolve/index.html Visual convolution Java Applet
- http://www.jhu.edu/~{}signals/discreteconv2/index.html Visual convolution Java Applet for discrete-time functions
- Lectures on Image Processing: A collection of 18 lectures in pdf format from Vanderbilt University. Lecture 7 is on 2-D convolution., by Alan Peters
  - http://archive.org/details/Lectures_on_Image_Processing
- Convolution Kernel Mask Operation Interactive tutorial
- Convolution at MathWorld
- Freeverb3 Impulse Response Processor: Opensource zero latency impulse response processor with VST plugins
- Stanford University CS 178 interactive Flash demo showing how spatial convolution works.
- A video lecture on the subject of convolution given by Salman Khan
- A Javascript interactive plot of the convolution with several functions



Gaussian blur can be used in order to obtain a smooth grayscale digital image of a halftone print

Chapter 9

Courant bracket

In a field of mathematics known as differential geometry, the **Courant bracket** is a generalization of the Lie bracket from an operation on the tangent bundle to an operation on the direct sum of the tangent bundle and the vector bundle of *p*-forms.

The case $p = 1$ was introduced by Theodore James Courant in his 1990 doctoral dissertation as a structure that bridges Poisson geometry and presymplectic geometry, based on work with his advisor Alan Weinstein. The twisted version of the Courant bracket was introduced in 2001 by Pavol Severa, and studied in collaboration with Weinstein.

Today a complex version of the p=1 Courant bracket plays a central role in the field of generalized complex geometry, introduced by Nigel Hitchin in 2002. Closure under the Courant bracket is the integrability condition of a generalized almost complex structure.

9.1 Definition

Let X and Y be vector fields on an N-dimensional real manifold M and let ξ and η be p-forms. Then X+ξ and Y+η are sections of the direct sum of the tangent bundle and the bundle of p-forms. The Courant bracket of X+ξ and Y+η is defined to be

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i(X)\eta - i(Y)\xi)$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X, d is the exterior derivative and i is the interior product.

9.2 Properties

The Courant bracket is antisymmetric but it does not satisfy the Jacobi identity for p greater than zero.

9.2.1 The Jacobi identity

However, at least in the case *p=1*, the Jacobiator, which measures a bracket's failure to satisfy the Jacobi identity, is an exact form. It is the exterior derivative of a form which plays the role of the Nijenhuis tensor in generalized complex geometry.

The Courant bracket is the antisymmetrization of the Dorfman bracket, which does satisfy a kind of Jacobi identity.

9.2.2 Symmetries

Like the Lie bracket, the Courant bracket is invariant under diffeomorphisms of the manifold M. It also enjoys an additional symmetry under the vector bundle automorphism

$$X + \xi \mapsto X + \xi + i(X)\alpha$$

where α is a closed *p+1*-form. In the *p=1* case, which is the relevant case for the geometry of flux compactifications in string theory, this transformation is known in the physics literature as a shift in the B field.

9.3 Dirac and generalized complex structures

The cotangent bundle, $\mathbf{T}^*$ of M is the bundle of differential one-forms. In the case *p*=1 the Courant bracket maps two sections of $\mathbf{T} \oplus \mathbf{T}^*$, the direct sum of the tangent and cotangent bundles, to another section of $\mathbf{T} \oplus \mathbf{T}^*$. The fibers of $\mathbf{T} \oplus \mathbf{T}^*$ admit inner products with signature (N,N) given by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

A linear subspace of $\mathbf{T} \oplus \mathbf{T}^*$ in which all pairs of vectors have zero inner product is said to be an isotropic subspace. The fibers of $\mathbf{T}\oplus\mathbf{T}^*$ are *2N*-dimensional and the maximal dimension of an isotropic subspace is *N*. An *N*-dimensional isotropic subspace is called a maximal isotropic subspace.

A Dirac structure is a maximally isotropic subbundle of $\mathbf{T} \oplus \mathbf{T}^*$ whose sections are closed under the Courant bracket. Dirac structures include as special cases symplectic structures, Poisson structures and foliated geometries.

A generalized complex structure is defined identically, but one tensors $\mathbf{T} \oplus \mathbf{T}^*$ by the complex numbers and uses the complex dimension in the above definitions and one imposes that the direct sum of the subbundle and its complex conjugate be the entire original bundle $(\mathbf{T} \oplus \mathbf{T}^*) \otimes \mathbf{C}$. Special cases of generalized complex structures include complex structure and a version of Kähler structure which includes the B-field.

9.4 Dorfman bracket

In 1987 Irene Dorfman introduced the Dorfman bracket [,]D, which like the Courant bracket provides an integrability condition for Dirac structures. It is defined by

$$[A, B]_D = [A, B] + d\langle A, B\rangle$$

The Dorfman bracket is not antisymmetric, but it is often easier to calculate with than the Courant bracket because it satisfies a Leibniz rule which resembles the Jacobi identity

$$[A, [B, C]_D]_D = [[A, B]_D, C]_D + [B, [A, C]_D]_D.$$

9.5 Courant algebroid

The Courant bracket does not satisfy the Jacobi identity and so it does not define a Lie algebroid, in addition it fails to satisfy the Lie algebroid condition on the anchor map. Instead it defines a more general structure introduced by Zhang-Ju Liu, Alan Weinstein and Ping Xu known as a Courant algebroid.

9.6 Twisted Courant bracket

9.6.1 Definition and properties

The Courant bracket may be twisted by a *(p+2)*-form *H*, by adding the interior product of the vector fields *X* and *Y* of *H*. It remains antisymmetric and invariant under the addition of the interior product with a *(p+1)*-form *B*. When *B* is not closed then this invariance is still preserved if one adds *dB* to the final *H*.

If *H* is closed then the Jacobiator is exact and so the twisted Courant bracket still defines a Courant algebroid. In string theory, *H* is interpreted as the Neveu-Schwarz 3-form.

9.6.2 *p=0*: Circle-invariant vector fields

When *p*=0 the Courant bracket reduces to the Lie bracket on a principal circle bundle over *M* with curvature given by the 2-form twist *H*. The bundle of 0-forms is the trivial bundle, and a section of the direct sum of the tangent bundle and the trivial bundle defines a circle invariant vector field on this circle bundle.

Concretely, a section of the sum of the tangent and trivial bundles is given by a vector field *X* and a function *f* and the Courant bracket is

$$[X+f,Y+g]=[X,Y]+Xg-Yf$$

which is just the Lie bracket of the vector fields

$$[X+f,Y+g]=[X+f\frac{\partial}{\partial\theta},Y+g\frac{\partial}{\partial\theta}]_{Lie}$$

where θ is a coordinate on the circle fiber. Note in particular that the Courant bracket satisfies the Jacobi identity in the case *p=0*.

9.6.3 Integral twists and gerbes

The curvature of a circle bundle always represents an integral cohomology class, the Chern class of the circle bundle. Thus the above geometric interpretation of the twisted *p=0* Courant bracket only exists when *H* represents an integral class. Similarly at higher values of *p* the twisted Courant brackets can be geometrically realized as untwisted Courant brackets twisted by gerbes when *H* is an integral cohomology class.

9.7 References

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Chapter 10

Cross product

This article is about the cross product of two vectors in three-dimensional Euclidean space. For other uses, see Cross product (disambiguation).

In mathematics and vector calculus, the **cross product** or **vector product** (occasionally **directed area product** to emphasize the geometric significance) is a binary operation on two vectors in three-dimensional space ($\mathbf{R}^3$) and is denoted by the symbol ×. Given two linearly independent vectors **a** and **b**, the cross product, **a** × **b**, is a vector that is perpendicular to both and therefore normal to the plane containing them. It has many applications in mathematics, physics, engineering, and computer programming. It should not be confused with dot product (projection product).

If two vectors have the same direction (or have the exact opposite direction from one another, i.e. are *not* linearly independent) or if either one has zero length, then their cross product is zero. More generally, the magnitude of the product equals the area of a parallelogram with the vectors for sides; in particular, the magnitude of the product of two perpendicular vectors is the product of their lengths. The cross product is anticommutative (i.e. **a** × **b** = −**b** × **a**) and is distributive over addition (i.e. **a** × (**b** + **c**) = **a** × **b** + **a** × **c**). The space $\mathbf{R}^3$ together with the cross product is an algebra over the real numbers, which is neither commutative nor associative, but is a Lie algebra with the cross product being the Lie bracket.

Like the dot product, it depends on the metric of Euclidean space, but unlike the dot product, it also depends on a choice of orientation or "handedness". The product can be generalized in various ways; it can be made independent of orientation by changing the result to pseudovector, or in arbitrary dimensions the exterior product of vectors can be used with a bivector or two-form result. Also, using the orientation and metric structure just as for the traditional 3-dimensional cross product, one can in n dimensions take the product of $n-1$ vectors to produce a vector perpendicular to all of them. But if the product is limited to non-trivial binary products with vector results, it exists only in three and seven dimensions.[1] If one adds the further requirement that the product be uniquely defined, then only the 3-dimensional cross product qualifies. (See § Generalizations, below, for other dimensions.)

10.1 Definition

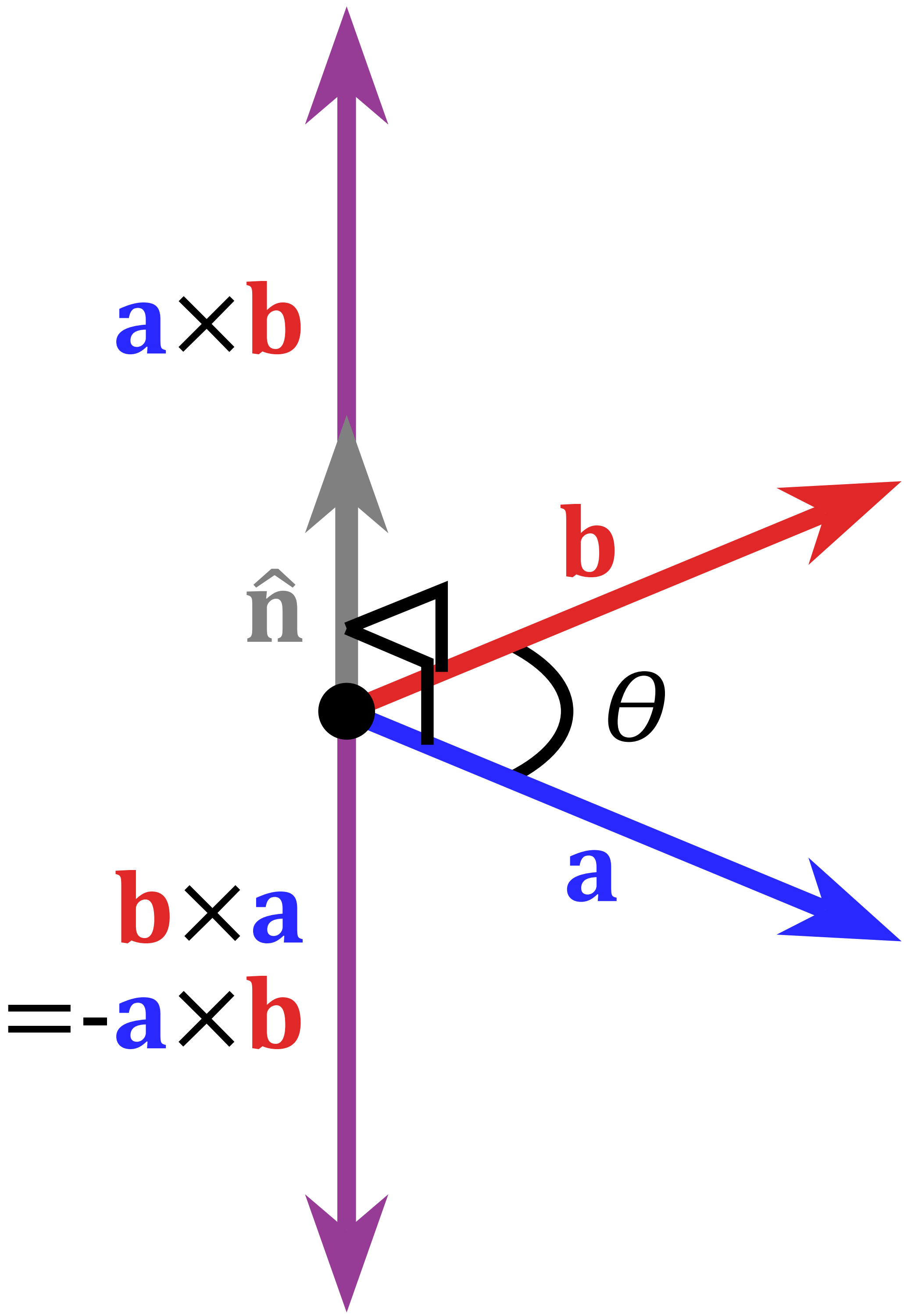
The cross product of two vectors **a** and **b** is defined only in three-dimensional space and is denoted by **a** × **b**. In physics, sometimes the notation **a** ∧ **b** is used,[2] though this is avoided in mathematics to avoid confusion with the exterior product.

The cross product **a** × **b** is defined as a vector **c** that is perpendicular to both **a** and **b**, with a direction given by the right-hand rule and a magnitude equal to the area of the parallelogram that the vectors span.

The cross product is defined by the formula[3][4]

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \; \mathbf{n}$$

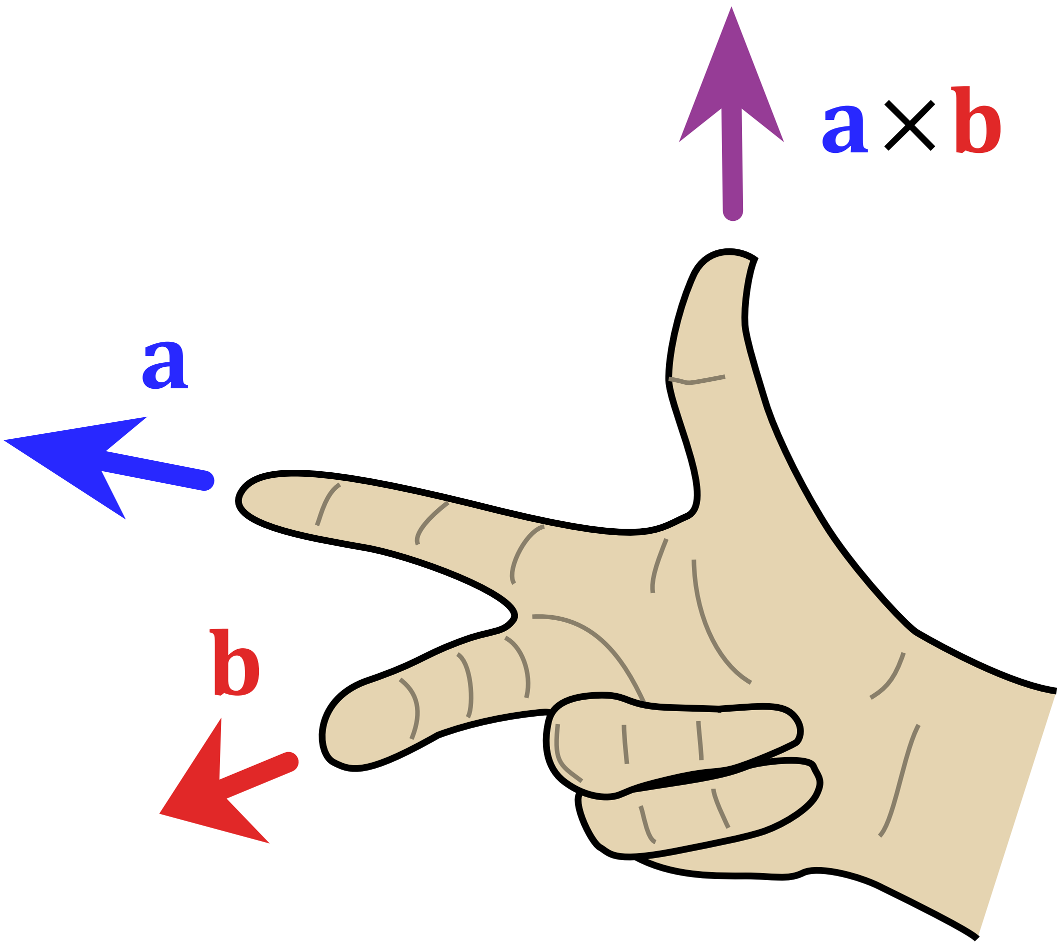
where θ is the angle between **a** and **b** in the plane containing them (hence, it is between 0° and 180°), ‖**a**‖ and ‖**b**‖ are the magnitudes of vectors **a** and **b**, and **n** is a unit vector perpendicular to the plane containing **a** and **b** in the



The cross-product in respect to a right-handed coordinate system

direction given by the right-hand rule (illustrated). If the vectors **a** and **b** are parallel (i.e., the angle θ between them is either 0° or 180°), by the above formula, the cross product of **a** and **b** is the zero vector **0**.

By convention, the direction of the vector **n** is given by the right-hand rule, where one simply points the forefinger of the right hand in the direction of **a** and the middle finger in the direction of **b**. Then, the vector **n** is coming out of



Finding the direction of the cross product by the right-hand rule

the thumb (see the picture on the right). Using this rule implies that the cross-product is anti-commutative, i.e., **b** × **a** = −(**a** × **b**). By pointing the forefinger toward **b** first, and then pointing the middle finger toward **a**, the thumb will be forced in the opposite direction, reversing the sign of the product vector.

Using the cross product requires the handedness of the coordinate system to be taken into account (as explicit in the definition above). If a left-handed coordinate system is used, the direction of the vector **n** is given by the left-hand rule and points in the opposite direction.

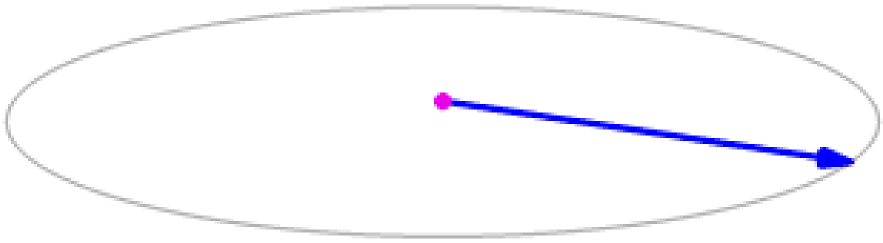
This, however, creates a problem because transforming from one arbitrary reference system to another (e.g., a mirror image transformation from a right-handed to a left-handed coordinate system), should not change the direction of **n**. The problem is clarified by realizing that the cross product of two vectors is not a (true) vector, but rather a *pseudovector*. See cross product and handedness for more detail.

10.2 Names

In 1881, Josiah Willard Gibbs, and independently Oliver Heaviside, introduced both the dot product and the cross product using a period (**a** . **b**) and an "x" (**a** x **b**), respectively, to denote them.[5]

In 1877, to emphasize the fact that the result of a dot product is a scalar while the result of a cross product is a vector, William Kingdon Clifford coined the alternative names **scalar product** and **vector product** for the two operations.[5] These alternative names are still widely used in the literature.

Both the cross notation (**a** × **b**) and the name **cross product** were possibly inspired by the fact that each scalar



The cross product $\mathbf{a} \times \mathbf{b}$ *(vertical, in purple) changes as the angle between the vectors* $\mathbf{a}$ *(blue) and* $\mathbf{b}$ *(red) changes. The cross product is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude* $\|\mathbf{a}\|\|\mathbf{b}\|$ *when they are perpendicular.*

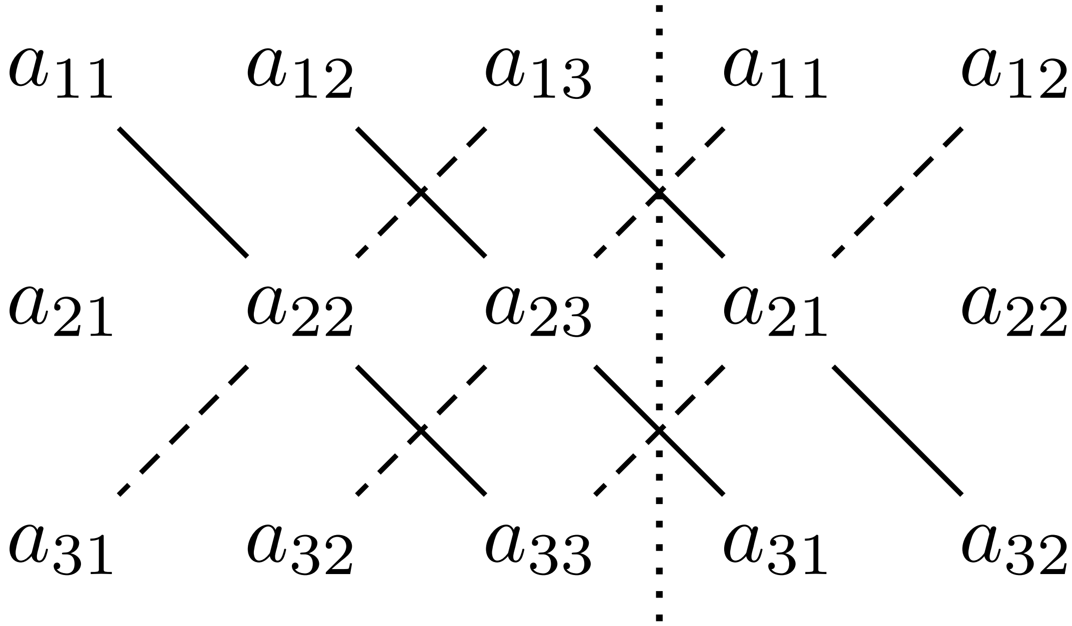
component of $\mathbf{a} \times \mathbf{b}$ is computed by multiplying non-corresponding components of $\mathbf{a}$ and $\mathbf{b}$. Conversely, a dot product $\mathbf{a} \cdot \mathbf{b}$ involves multiplications between corresponding components of $\mathbf{a}$ and $\mathbf{b}$. As explained below, the cross product can be expressed in the form of a determinant of a special 3×3 matrix. According to Sarrus' rule, this involves multiplications between matrix elements identified by crossed diagonals.

10.3 Computing the cross product

10.3.1 Coordinate notation

The standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy the following equalities in a right hand coordinate system:

$$\mathbf{i} = \mathbf{j} \times \mathbf{k}$$
$$\mathbf{j} = \mathbf{k} \times \mathbf{i}$$
$$\mathbf{k} = \mathbf{i} \times \mathbf{j}$$



According to Sarrus' rule, the determinant of a 3×3 matrix involves multiplications between matrix elements identified by crossed diagonals

which imply, by the anticommutativity of the cross product, that

$$\begin{aligned}\mathbf{k}\times\mathbf{j} &= -\mathbf{i}\\ \mathbf{i}\times\mathbf{k} &= -\mathbf{j}\\ \mathbf{j}\times\mathbf{i} &= -\mathbf{k}\end{aligned}$$

The definition of the cross product also implies that

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ (the zero vector).

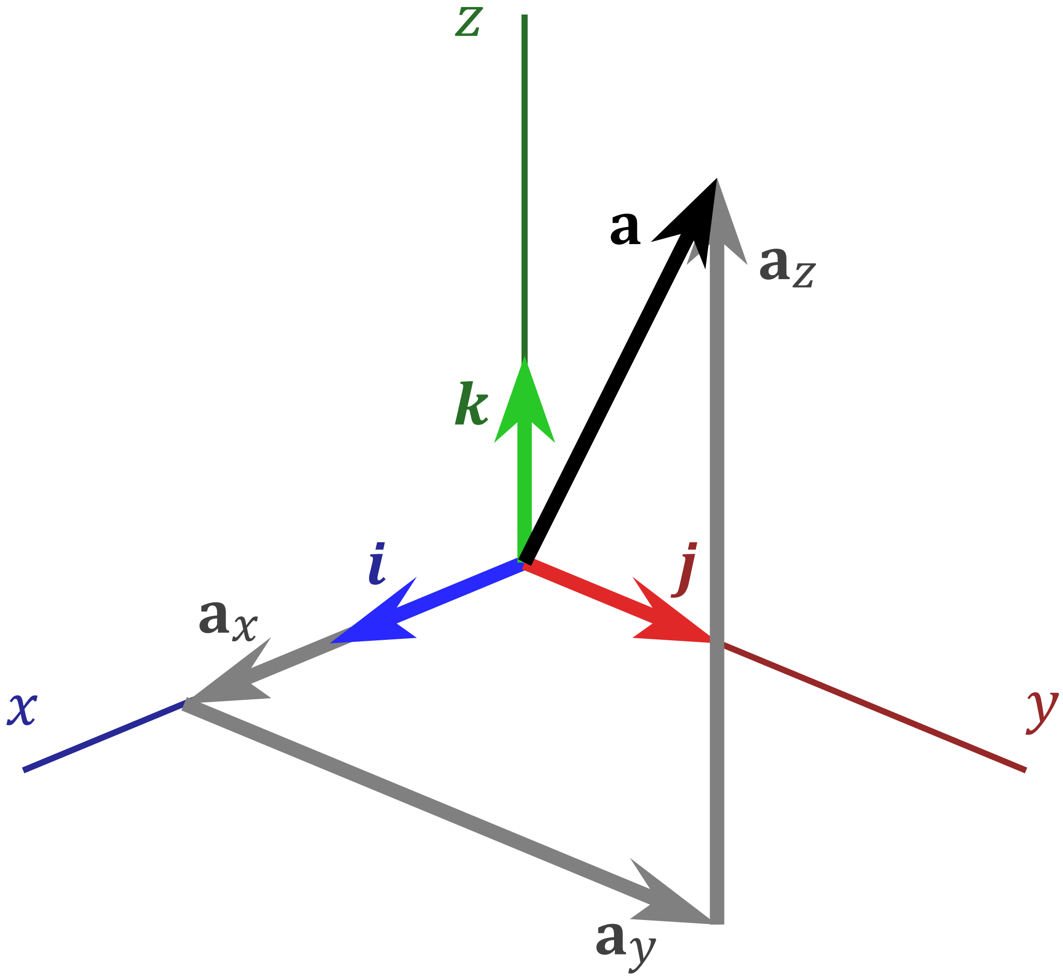
These equalities, together with the distributivity and linearity of the cross product (but both do not follow easily from the definition given above), are sufficient to determine the cross product of any two vectors **u** and **v**. Each vector can be defined as the sum of three orthogonal components parallel to the standard basis vectors:

$$\begin{aligned}\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}\\ \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\end{aligned}$$

Their cross product **u** × **v** can be expanded using distributivity:

$$\begin{aligned}\mathbf{u}\times\mathbf{v} = {} & (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})\\ = {} & u_1v_1(\mathbf{i}\times\mathbf{i}) + u_1v_2(\mathbf{i}\times\mathbf{j}) + u_1v_3(\mathbf{i}\times\mathbf{k}) +\\ & u_2v_1(\mathbf{j}\times\mathbf{i}) + u_2v_2(\mathbf{j}\times\mathbf{j}) + u_2v_3(\mathbf{j}\times\mathbf{k}) +\\ & u_3v_1(\mathbf{k}\times\mathbf{i}) + u_3v_2(\mathbf{k}\times\mathbf{j}) + u_3v_3(\mathbf{k}\times\mathbf{k})\end{aligned}$$

This can be interpreted as the decomposition of **u** × **v** into the sum of nine simpler cross products involving vectors aligned with **i**, **j**, or **k**. Each one of these nine cross products operates on two vectors that are easy to handle as they are either parallel or orthogonal to each other. From this decomposition, by using the above-mentioned equalities and collecting similar terms, we obtain:



*Standard basis vectors (**i**, **j**, **k**, also denoted $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) and vector components of **a** (**a**x, **a**y, **a**z, also denoted $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$)*

$$\begin{aligned}\mathbf{u}\times\mathbf{v} = {} & u_1v_1\mathbf{0} + u_1v_2\mathbf{k} - u_1v_3\mathbf{j} - \\ & u_2v_1\mathbf{k} - u_2v_2\mathbf{0} + u_2v_3\mathbf{i} + \\ & u_3v_1\mathbf{j} - u_3v_2\mathbf{i} - u_3v_3\mathbf{0} \\ = {} & (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

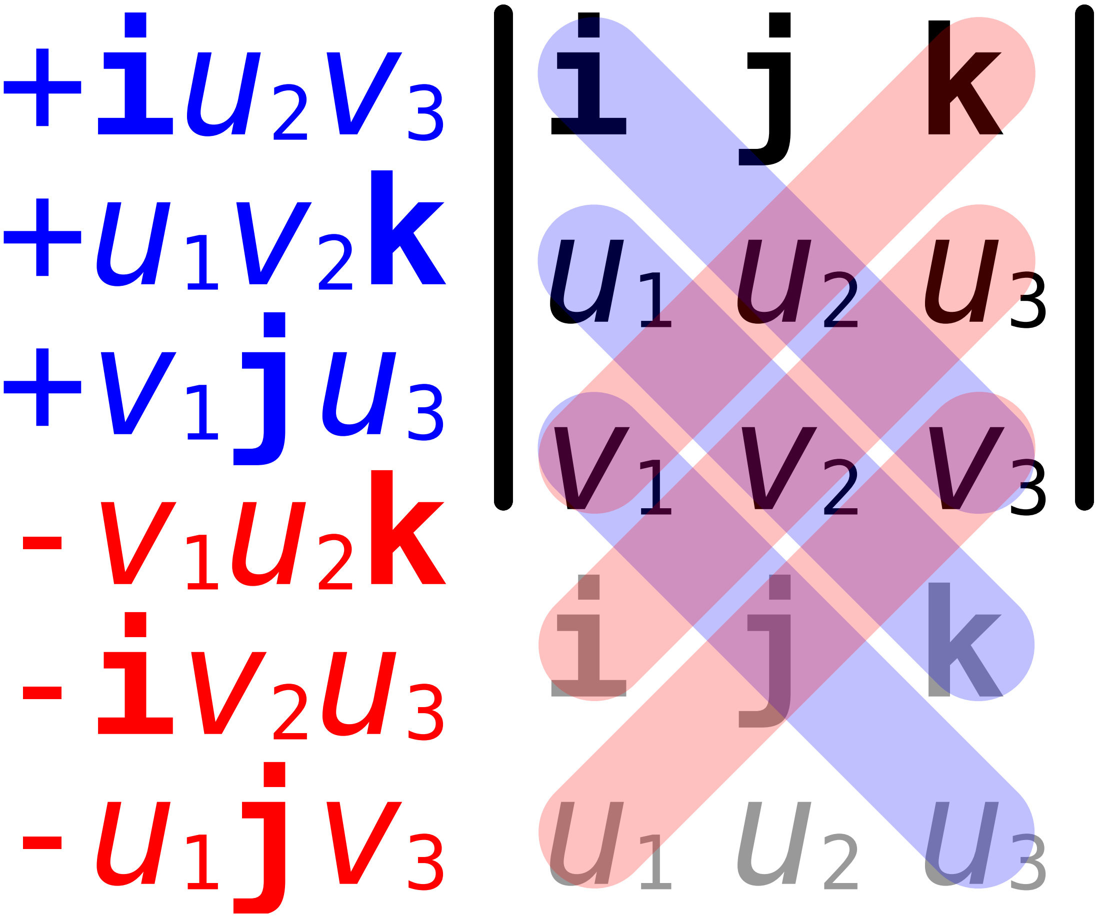
meaning that the three scalar components of the resulting vector $\mathbf{s} = s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k} = \mathbf{u}\times\mathbf{v}$ are

$$\begin{aligned}s_1 &= u_2v_3 - u_3v_2\\ s_2 &= u_3v_1 - u_1v_3\\ s_3 &= u_1v_2 - u_2v_1\end{aligned}$$

Using column vectors, we can represent the same result as follows:

$$\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} = \begin{pmatrix}u_2v_3 - u_3v_2\\ u_3v_1 - u_1v_3\\ u_1v_2 - u_2v_1\end{pmatrix}$$

10.3.2 Matrix notation



*Use of Sarrus' rule to find the cross product of **u** and **v***

The cross product can also be expressed as the formal[note 1] determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

This determinant can be computed using Sarrus' rule or cofactor expansion. Using Sarrus' rule, it expands to

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 \mathbf{i} + u_3 v_1 \mathbf{j} + u_1 v_2 \mathbf{k}) - (u_3 v_2 \mathbf{i} + u_1 v_3 \mathbf{j} + u_2 v_1 \mathbf{k}) \\ &= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}. \end{aligned}$$

Using cofactor expansion along the first row instead, it expands to[6]

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

which gives the components of the resulting vector directly.

10.4 Properties

10.4.1 Geometric meaning

See also: Triple product
The magnitude of the cross product can be interpreted as the positive area of the parallelogram having **a** and **b** as

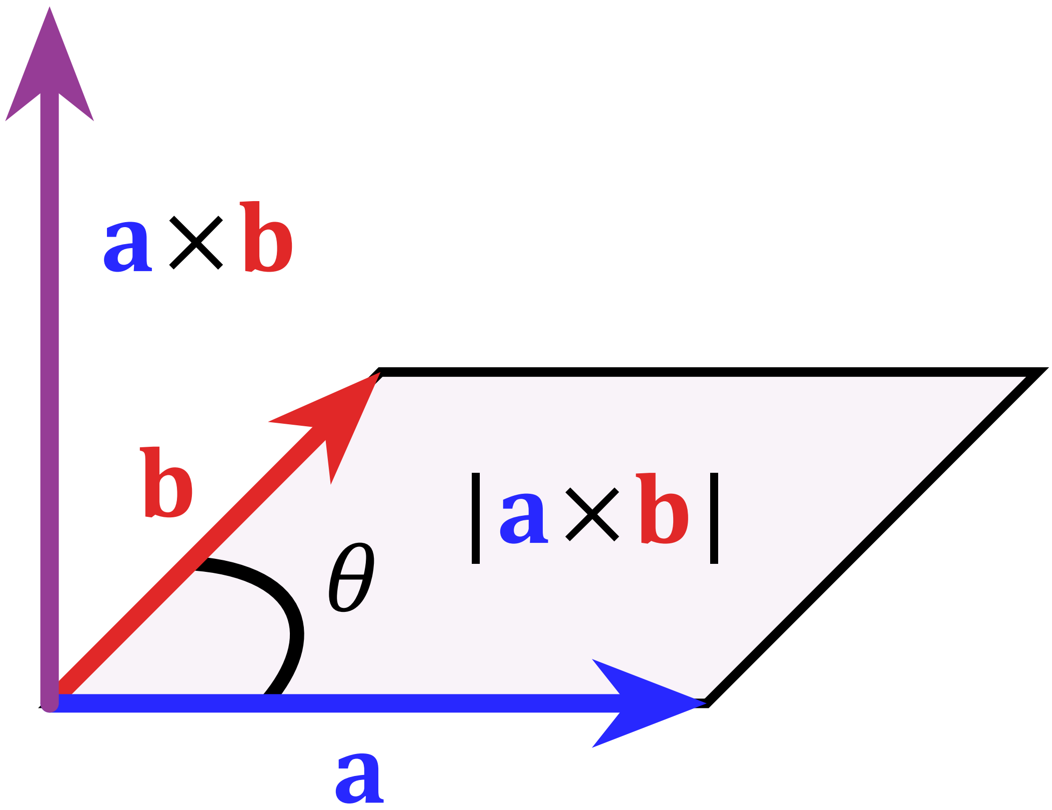


Figure 1. The area of a parallelogram as the magnitude of a cross product

sides (see Figure 1):

$A = \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta.$

Indeed, one can also compute the volume V of a parallelepiped having **a**, **b** and **c** as edges by using a combination of a cross product and a dot product, called scalar triple product (see Figure 2):

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$

Since the result of the scalar triple product may be negative, the volume of the parallelepiped is given by its absolute value. For instance,

$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$

Because the magnitude of the cross product goes by the sine of the angle between its arguments, the cross product can be thought of as a measure of *perpendicularity* in the same way that the dot product is a measure of *parallelism*. Given two unit vectors, their cross product has a magnitude of 1 if the two are perpendicular and a magnitude of zero if the two are parallel. The converse is true for the dot product of two unit vectors.

Unit vectors enable two convenient identities: the dot product of two unit vectors yields the cosine (which may be positive or negative) of the angle between the two unit vectors. The magnitude of the cross product of the two unit vectors yields the sine (which will always be positive).

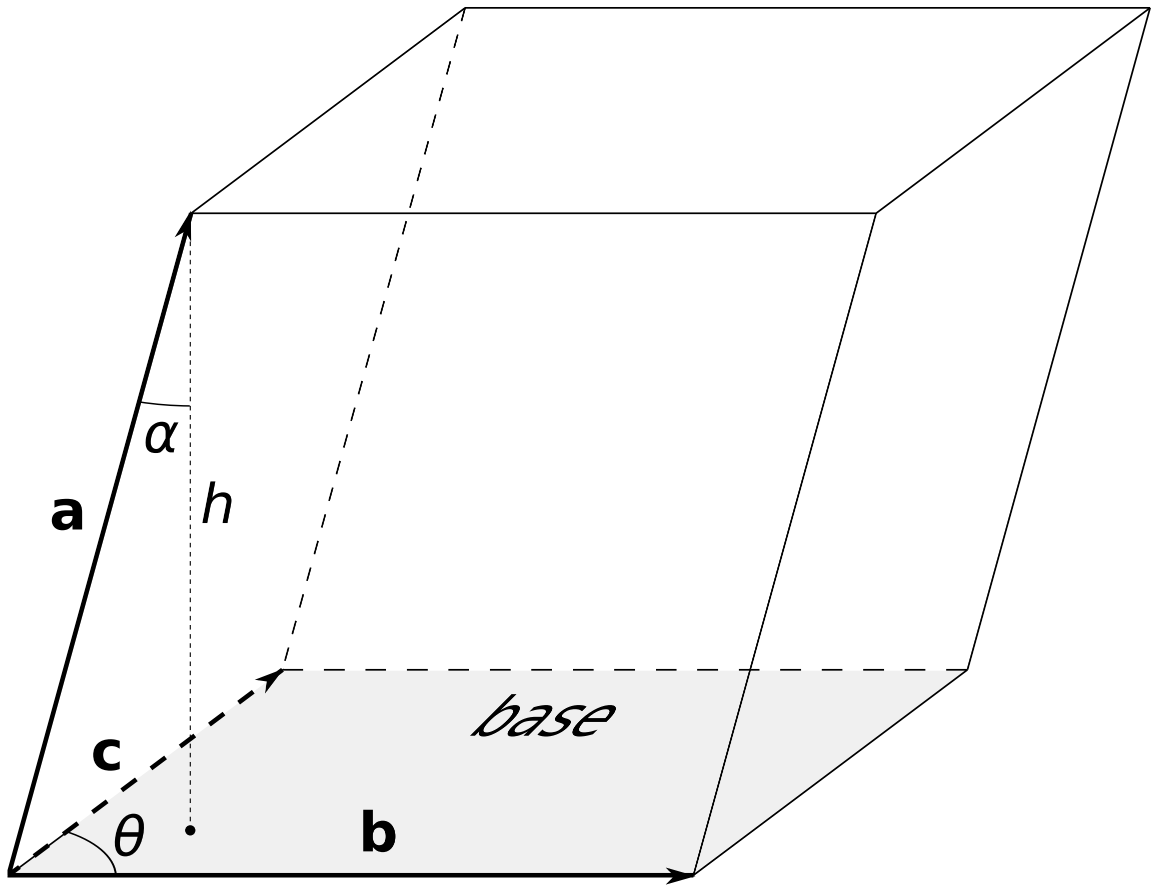
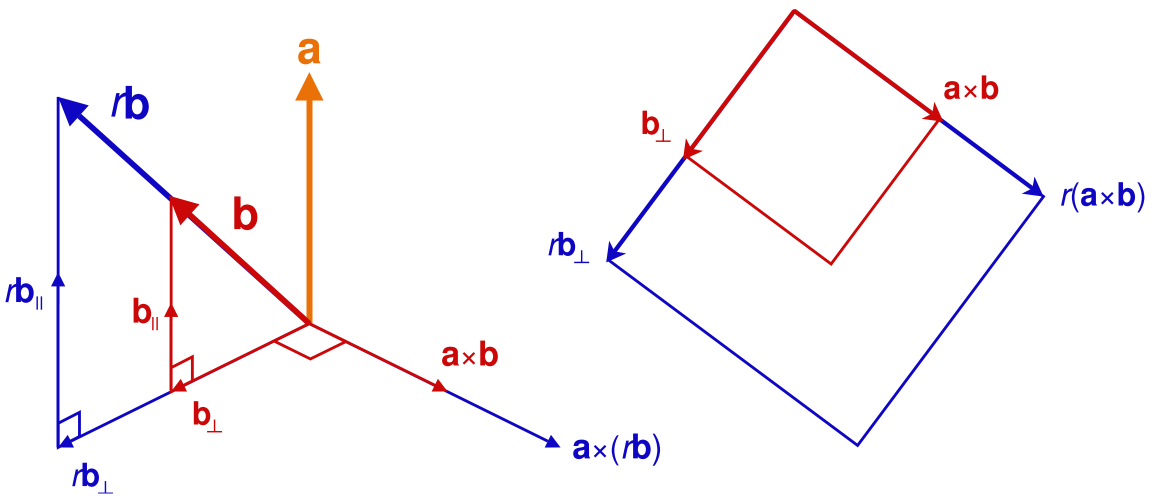


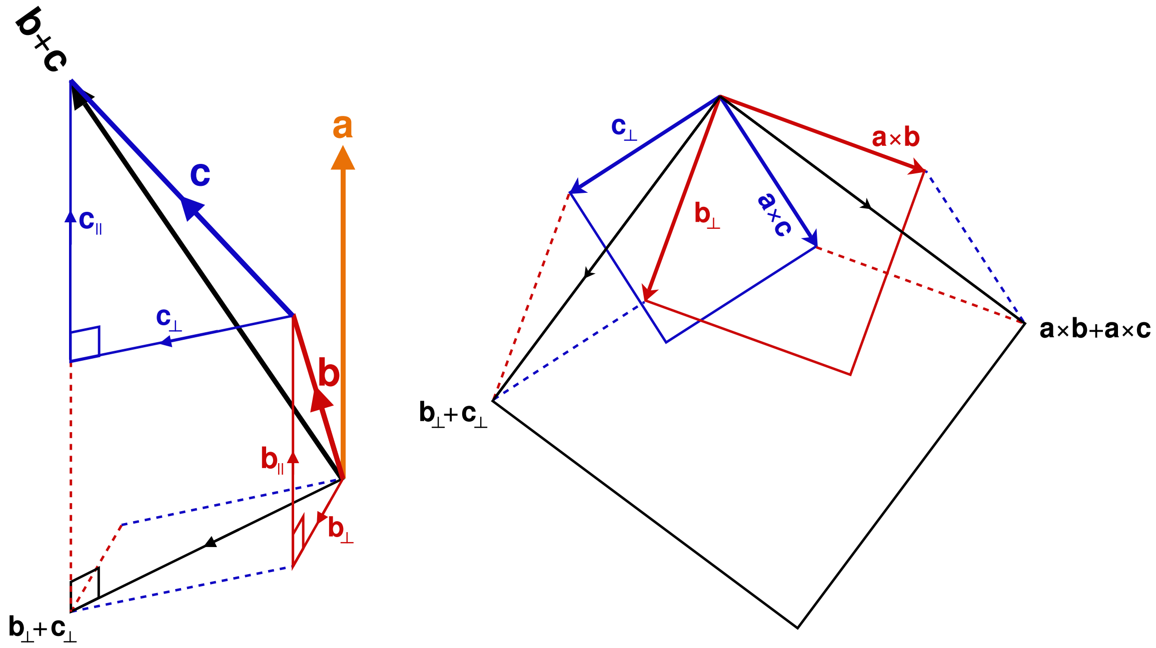
Figure 2. Three vectors defining a parallelepiped

10.4.2 Algebraic properties

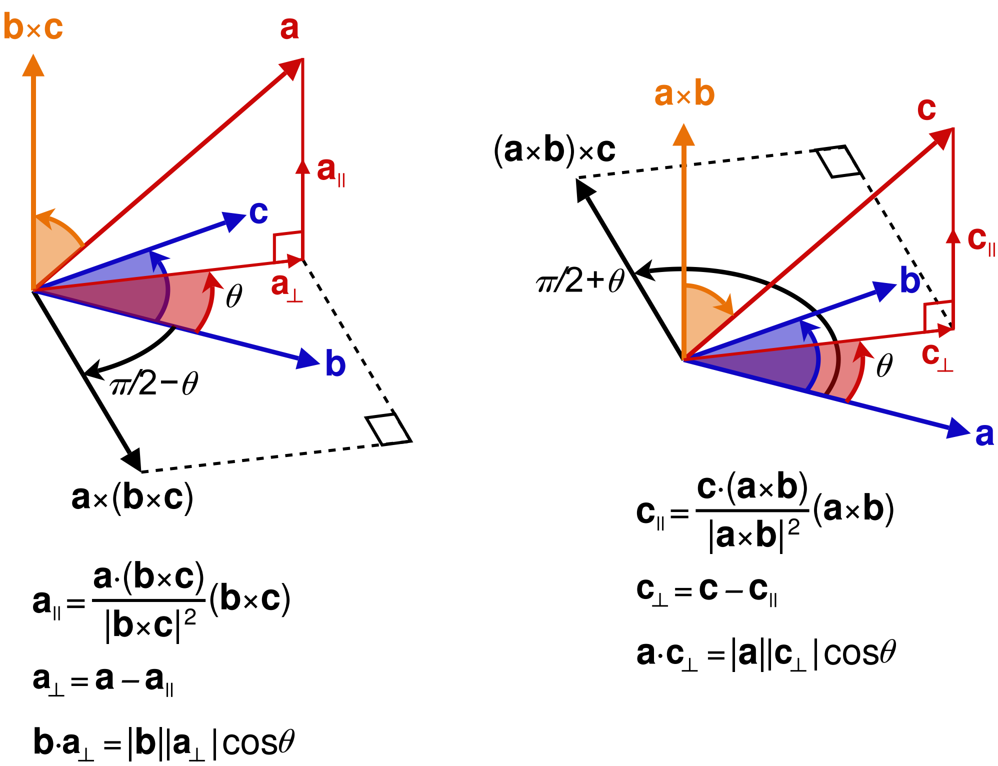


*Cross product scalar multiplication. **Left:** Decomposition of **b** into components parallel and perpendicular to **a**. Right: Scaling of the perpendicular components by a positive real number* r *(if negative, **b** and the cross product are reversed).*

- If the cross product of two vectors is the zero vector (i.e. $\mathbf{a} \times \mathbf{b} = \mathbf{0}$), then either one or both of the inputs is the zero vector, ($\mathbf{a} = \mathbf{0}$ and/or $\mathbf{b} = \mathbf{0}$) or else they are parallel or antiparallel ($\mathbf{a} \parallel \mathbf{b}$) so that the sine of the angle between them is zero ($\theta = 0°$ or $\theta = 180°$ and $\sin\theta = 0$).
- The self cross product of a vector is the zero vector, i.e., $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.



*Cross product distributivity over vector addition. **Left:** The vectors **b** and **c** are resolved into parallel and perpendicular components to **a**. **Right:** The parallel components vanish in the cross product, only the perpendicular components shown in the plane perpendicular to **a** remain.*[7]



*The two nonequivalent triple cross products of three vectors **a**, **b**, **c**. In each case, two vectors define a plane, the other is out of the plane and can be split into parallel and perpendicular components to the cross product of the vectors defining the plane. These components can be found by vector projection and rejection. The triple product is in the plane and is rotated as shown.*

- The cross product is anticommutative,

$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}),$

- distributive over addition,

$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$

- and compatible with scalar multiplication so that

$(r\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (r\mathbf{b}) = r(\mathbf{a} \times \mathbf{b}).$

- It is not associative, but satisfies the Jacobi identity:

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$

Distributivity, linearity and Jacobi identity show that the $\mathbf{R}^3$ vector space together with vector addition and the cross product forms a Lie algebra, the Lie algebra of the real orthogonal group in 3 dimensions, SO(3).

- The cross product does not obey the cancellation law: that is, $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ does not imply $\mathbf{b} = \mathbf{c}$, but only that:

$$\begin{aligned} \mathbf{0} &= (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{c}) \\ &= \mathbf{a} \times (\mathbf{b} - \mathbf{c}). \end{aligned}$$

From the definition of the cross product, the angle between $\mathbf{a}$ and $\mathbf{b} - \mathbf{c}$ must be zero, and these vectors must be parallel. That is, they are related by a scale factor t, leading to:

$\mathbf{c} = \mathbf{b} + t\mathbf{a},$

for some scalar t.

- If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, for non-zero vector $\mathbf{a}$, then $\mathbf{b} = \mathbf{c}$, as

$\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$

$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0,$

so $\mathbf{b} - \mathbf{c}$ is both parallel and perpendicular to the non-zero vector a, something that is only possible if $\mathbf{b} - \mathbf{c} = \mathbf{0}$ so they are identical.

- From the geometrical definition, the cross product is invariant under proper rotations about the axis defined by $\mathbf{a} \times \mathbf{b}$. In formulae:

  $(R\mathbf{a}) \times (R\mathbf{b}) = R(\mathbf{a} \times \mathbf{b})$, where R is a rotation matrix with $\det(R) = 1$.

More generally, the cross product obeys the following identity under matrix transformations:

$(M\mathbf{a}) \times (M\mathbf{b}) = (\det M) M^{-T} (\mathbf{a} \times \mathbf{b}) = \operatorname{cof} M (\mathbf{a} \times \mathbf{b})$

where M is a 3-by-3 matrix and M^{-T} is the transpose of the inverse and cof is the cofactor matrix. It can be readily seen how this formula reduces to the former one if M is a rotation matrix.

- The cross product of two vectors lies in the null space of the 2×3 matrix with the vectors as rows:

$\mathbf{a} \times \mathbf{b} \in NS\left(\begin{bmatrix}\mathbf{a} \\ \mathbf{b}\end{bmatrix}\right).$

- For the sum of two cross products, the following identity holds:

$\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{d} = (\mathbf{a} - \mathbf{c}) \times (\mathbf{b} - \mathbf{d}) + \mathbf{a} \times \mathbf{d} + \mathbf{c} \times \mathbf{b}.$

10.4.3 Differentiation

Main article: Vector-valued function § Derivative and vector multiplication

The product rule of differential calculus applies to any bilinear operation, and therefore also to the cross product:

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}$$

where **a** and **b** are vectors that depend on the real variable *t*.

10.4.4 Triple product expansion

Main article: Triple product

The cross product is used in both forms of the triple product. The scalar triple product of three vectors is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}),$$

It is the signed volume of the parallelepiped with edges **a**, **b** and **c** and as such the vectors can be used in any order that's an even permutation of the above ordering. The following therefore are equal:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),$$

The vector triple product is the cross product of a vector with the result of another cross product, and is related to the dot product by the following formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

The mnemonic "BAC minus CAB" is used to remember the order of the vectors in the right hand member. This formula is used in physics to simplify vector calculations. A special case, regarding gradients and useful in vector calculus, is

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{f}) &= \nabla(\nabla \cdot \mathbf{f}) - (\nabla \cdot \nabla)\mathbf{f} \\ &= \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}, \end{aligned}$$

where ∇^2 is the vector Laplacian operator.

Other identities relate the cross product to the scalar triple product:

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}))\mathbf{a}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{b}^T((\mathbf{c}^T\mathbf{a})I - \mathbf{c}\mathbf{a}^T)\mathbf{d} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

where I is the identity matrix.

10.4.5 Alternative formulation

The cross product and the dot product are related by:

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

The right-hand side is the Gram determinant of **a** and **b**, the square of the area of the parallelogram defined by the vectors. This condition determines the magnitude of the cross product. Namely, since the dot product is defined, in terms of the angle θ between the two vectors, as:

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta,$$

the above given relationship can be rewritten as follows:

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\left(1-\cos^2\theta\right).$$

Invoking the Pythagorean trigonometric identity one obtains:

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\sin\theta|\,,$$

which is the magnitude of the cross product expressed in terms of θ, equal to the area of the parallelogram defined by **a** and **b** (see definition above).

The combination of this requirement and the property that the cross product be orthogonal to its constituents **a** and **b** provides an alternative definition of the cross product.[8]

10.4.6 Lagrange's identity

The relation:

$$\|\mathbf{a}\times\mathbf{b}\|^2 = \det\begin{bmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b}\end{bmatrix} = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2.$$

can be compared with another relation involving the right-hand side, namely Lagrange's identity expressed as:[9]

$$\sum_{1\le i<j\le n}\left(a_ib_j-a_jb_i\right)^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2\,,$$

where **a** and **b** may be n-dimensional vectors. This also shows that the Riemannian volume form for surfaces is exactly the surface element from vector calculus. In the case where $n = 3$, combining these two equations results in the expression for the magnitude of the cross product in terms of its components:[10]

$$|\mathbf{a}\times\mathbf{b}|^2 = \sum_{1\le i<j\le 3}\left(a_ib_j-a_jb_i\right)^2 = (a_1b_2-b_1a_2)^2+(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2\,.$$

The same result is found directly using the components of the cross-product found from:

$$\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}.$$

In $\mathbf{R}^3$ Lagrange's equation is a special case of the multiplicativity |**vw**| = |**v**||**w**| of the norm in the quaternion algebra.

It is a special case of another formula, also sometimes called Lagrange's identity, which is the three dimensional case of the Binet-Cauchy identity:[11][12]

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).$$

If **a** = **c** and **b** = **d** this simplifies to the formula above.

10.4.7 Infinitesimal generators of rotations

The cross product conveniently describes the infinitesimal generators of rotations in $\mathbf{R}^3$. Specifically, if $\mathbf{n}$ is a unit vector in $\mathbf{R}^3$ and $R(\varphi, \mathbf{n})$ denotes a rotation about the axis through the origin specified by $\mathbf{n}$, with angle φ (measured in radians, counterclockwise when viewed from the tip of $\mathbf{n}$), then

$$\left.\frac{d}{d\phi}\right|_{\phi=0} R(\phi, \boldsymbol{n})\boldsymbol{x} = \boldsymbol{n} \times \boldsymbol{x}$$

for every vector $\mathbf{x}$ in $\mathbf{R}^3$. The cross product with $\mathbf{n}$ therefore describes the infinitesimal generator of the rotations about $\mathbf{n}$. These infinitesimal generators form the Lie algebra **so(3)** of the rotation group SO(3), and we obtain the result that the Lie algebra $\mathbf{R}^3$ with cross product is isomorphic to the Lie algebra **so(3)**.

10.5 Alternative ways to compute the cross product

10.5.1 Conversion to matrix multiplication

The vector cross product also can be expressed as the product of a skew-symmetric matrix and a vector:[11]

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = [\mathbf{b}]_{\times}^{\mathrm{T}} \mathbf{a} = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where superscript $^{\mathrm{T}}$ refers to the transpose operation, and $[\mathbf{a}]_{\times}$ is defined by:

$$[\mathbf{a}]_{\times} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

It should be noted that $[\mathbf{a}]_{\times}$ is a singular matrix where $\mathbf{a}$ is its (right and left) null-vector.

Also, if $\mathbf{a}$ is itself a cross product:

$$\mathbf{a} = \mathbf{c} \times \mathbf{d}$$

then

$$[\mathbf{a}]_{\times} = \mathbf{d}\mathbf{c}^{\mathrm{T}} - \mathbf{c}\mathbf{d}^{\mathrm{T}}.$$

This result can be generalized to higher dimensions using geometric algebra. In particular in any dimension bivectors can be identified with skew-symmetric matrices, so the product between a skew-symmetric matrix and vector is equivalent to the grade-1 part of the product of a bivector and vector. In three dimensions bivectors are dual to vectors so the product is equivalent to the cross product, with the bivector instead of its vector dual. In higher dimensions the product can still be calculated but bivectors have more degrees of freedom and are not equivalent to vectors.

This notation is also often much easier to work with, for example, in epipolar geometry.

From the general properties of the cross product follows immediately that

$[\mathbf{a}]_\times \mathbf{a} = \mathbf{0}$ and $\mathbf{a}^\mathrm{T} [\mathbf{a}]_\times = \mathbf{0}$

and from fact that [**a**]× is skew-symmetric it follows that

$$\mathbf{b}^\mathrm{T} [\mathbf{a}]_\times \mathbf{b} = 0.$$

The above-mentioned triple product expansion (bac–cab rule) can be easily proven using this notation.

As mentioned above, the Lie algebra $\mathbf{R}^3$ with cross product is isomorphic to the Lie algebra **so(3)**, whose elements can be identified with the 3×3 skew-symmetric matrices. The map **a** → [**a**]× provides an isomorphism between $\mathbf{R}^3$ and **so(3)**. Under this map, the cross product of 3-vectors corresponds to the commutator of 3x3 skew-symmetric matrices.

10.5.2 Index notation for tensors

The cross product can alternatively be defined in terms of the Levi-Civita symbol ε_{ijk} and a dot product η^{mi} (= δ^{mi} for an orthonormal basis), which are useful in converting vector notation for tensor applications:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \Leftrightarrow\ c^m = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \eta^{mi} \varepsilon_{ijk} a^j b^k$$

where the indices i,j,k correspond to vector components. This characterization of the cross product is often expressed more compactly using the Einstein summation convention as

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \Leftrightarrow\ c^m = \eta^{mi} \varepsilon_{ijk} a^j b^k$$

in which repeated indices are summed over the values 1 to 3. Note that this representation is another form of the skew-symmetric representation of the cross product:

$$\eta^{mi} \varepsilon_{ijk} a^j = [\mathbf{a}]_\times.$$

In classical mechanics: representing the cross-product by using the Levi-Civita symbol can cause mechanical symmetries to be obvious when physical systems are isotropic. (An example: consider a particle in a Hooke's Law potential in three-space, free to oscillate in three dimensions; none of these dimensions are "special" in any sense, so symmetries lie in the cross-product-represented angular momentum, which are made clear by the abovementioned Levi-Civita representation).

10.5.3 Mnemonic

"Xyzzy (mnemonic)" redirects here. For other uses, see Xyzzy.

The word "xyzzy" can be used to remember the definition of the cross product.

If

$$\mathbf{a} = \mathbf{b} \times \mathbf{c}$$

where:

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

then:

$$a_x = b_y c_z - b_z c_y$$

$$a_y = b_z c_x - b_x c_z$$

$$a_z = b_x c_y - b_y c_x.$$

The second and third equations can be obtained from the first by simply vertically rotating the subscripts, $x \to y \to z \to x$. The problem, of course, is how to remember the first equation, and two options are available for this purpose: either to remember the relevant two diagonals of Sarrus's scheme (those containing ***i***), or to remember the xyzzy sequence.

Since the first diagonal in Sarrus's scheme is just the main diagonal of the above-mentioned 3×3 matrix, the first three letters of the word xyzzy can be very easily remembered.

10.5.4 Cross visualization

Similarly to the mnemonic device above, a "cross" or X can be visualized between the two vectors in the equation. This may be helpful for remembering the correct cross product formula.

If

$$\mathbf{a} = \mathbf{b} \times \mathbf{c}$$

then:

$$\mathbf{a} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}.$$

If we want to obtain the formula for a_x we simply drop the b_x and c_x from the formula, and take the next two components down:

$$a_x = \begin{bmatrix} b_y \\ b_z \end{bmatrix} \times \begin{bmatrix} c_y \\ c_z \end{bmatrix}.$$

It should be noted that when doing this for a_y the next two elements down should "wrap around" the matrix so that after the z component comes the x component. For clarity, when performing this operation for a_y , the next two components should be z and x (in that order). While for a_z the next two components should be taken as x and y.

$$a_y = \begin{bmatrix} b_z \\ b_x \end{bmatrix} \times \begin{bmatrix} c_z \\ c_x \end{bmatrix}, a_z = \begin{bmatrix} b_x \\ b_y \end{bmatrix} \times \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

For a_x then, if we visualize the cross operator as pointing from an element on the left to an element on the right, we can take the first element on the left and simply multiply by the element that the cross points to in the right hand matrix. We then subtract the next element down on the left, multiplied by the element that the cross points to here as well. This results in our a_x formula –

$$a_x = b_y c_z - b_z c_y.$$

We can do this in the same way for a_y and a_z to construct their associated formulas.

10.6 Applications

The cross product has applications in various contexts: e.g. it is used in computational geometry, physics and engineering. A non-exhaustive list of examples follows.

10.6.1 Computational geometry

The cross product appears in the calculation of the distance of two skew lines (lines not in the same plane) from each other in three-dimensional space.

The cross product can be used to calculate the normal for a triangle or polygon, an operation frequently performed in computer graphics. For example, the winding of a polygon (clockwise or anticlockwise) about a point within the polygon can be calculated by triangulating the polygon (like spoking a wheel) and summing the angles (between the spokes) using the cross product to keep track of the sign of each angle.

In computational geometry of the plane, the cross product is used to determine the sign of the acute angle defined by three points $p_1=(x_1,y_1)$, $p_2=(x_2,y_2)$ and $p_3=(x_3,y_3)$. It corresponds to the direction of the cross product of the two coplanar vectors defined by the pairs of points p_1,p_2 and p_1,p_3 , i.e., by the sign of the expression $P=(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)$. In the “right-handed” coordinate system, if the result is 0, the points are collinear; if it is positive, the three points constitute a positive angle of rotation around p_1 from p_2 to p_3 , otherwise a negative angle. From another point of view, the sign of P tells whether p_3 lies to the left or to the right of line p_1,p_2 .

The cross product is used in calculating the volume of a polyhedron such as a tetrahedron or parallelepiped.

10.6.2 Angular momentum and torque

The angular momentum $\mathbf{L}$ of a particle about a given origin is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{r}$ is the position vector of the particle relative to the origin, $\mathbf{p}$ is the linear momentum of the particle.

In the same way, the moment $\mathbf{M}$ of a force $\mathbf{F_B}$ applied at point B around point A is given as:

$$\mathbf{M}_{\mathrm{A}} = \mathbf{r}_{\mathrm{AB}} \times \mathbf{F}_{\mathrm{B}}$$

In mechanics the *moment of a force* is also called *torque* and written as τ

Since position $\mathbf{r}$, linear momentum $\mathbf{p}$ and force $\mathbf{F}$ are all *true* vectors, both the angular momentum $\mathbf{L}$ and the moment of a force $\mathbf{M}$ are *pseudovectors* or *axial vectors*.

10.6.3 Rigid body

The cross product frequently appears in the description of rigid motions. Two points *P* and *Q* on a rigid body can be related by:

$$\mathbf{v}_P - \mathbf{v}_Q = \omega \times (\mathbf{r}_P - \mathbf{r}_Q)$$

where $\mathbf{r}$ is the point’s position, $\mathbf{v}$ is its velocity and ω is the body’s angular velocity.

Since position $\mathbf{r}$ and velocity $\mathbf{v}$ are *true* vectors, the angular velocity ω is a *pseudovector* or *axial vector*.

10.6.4 Lorentz force

See also: Lorentz force

The cross product is used to describe the Lorentz force experienced by a moving electric charge q_e :

$$\mathbf{F} = q_e \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

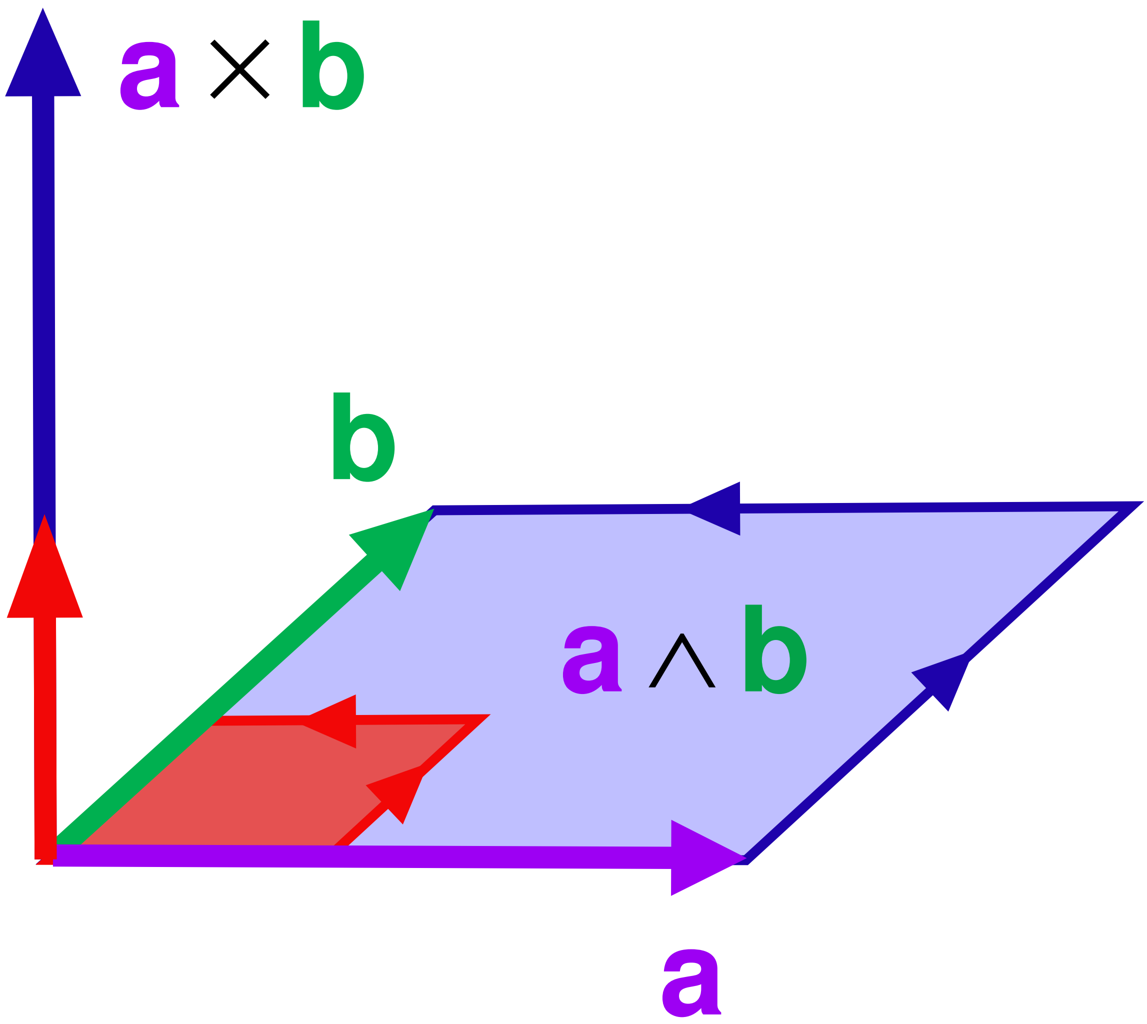
Since velocity **v** , force **F** and electric field **E** are all *true* vectors, the magnetic field **B** is a *pseudovector*.

10.6.5 Other

In vector calculus, the cross product is used to define the formula for the vector operator curl.

The trick of rewriting a cross product in terms of a matrix multiplication appears frequently in epipolar and multi-view geometry, in particular when deriving matching constraints.

10.7 Cross product as an exterior product



The cross product in relation to the exterior product. In red are the orthogonal unit vector, and the "parallel" unit bivector.

The cross product can be viewed in terms of the exterior product. This view allows for a natural geometric interpretation of the cross product. In exterior algebra the exterior product (or wedge product) of two vectors is a bivector. A bivector is an oriented plane element, in much the same way that a vector is an oriented line element. Given two vectors a and b, one can view the bivector $a \wedge b$ as the oriented parallelogram spanned by a and b. The cross product is then obtained by taking the Hodge dual of the bivector $a \wedge b$, mapping 2-vectors to vectors:

$a \times b = *(a \wedge b)$.

This can be thought of as the oriented multi-dimensional element "perpendicular" to the bivector. Only in three dimensions is the result an oriented line element – a vector – whereas, for example, in 4 dimensions the Hodge dual of a bivector is two-dimensional – another oriented plane element. So, only in three dimensions is the cross product of a and b the vector dual to the bivector $a \wedge b$: it is perpendicular to the bivector, with orientation dependent on the coordinate system's handedness, and has the same magnitude relative to the unit normal vector as $a \wedge b$ has relative to the unit bivector; precisely the properties described above.

10.8 Cross product and handedness

When measurable quantities involve cross products, the *handedness* of the coordinate systems used cannot be arbitrary. However, when physics laws are written as equations, it should be possible to make an arbitrary choice of the coordinate system (including handedness). To avoid problems, one should be careful to never write down an equation where the two sides do not behave equally under all transformations that need to be considered. For example, if one side of the equation is a cross product of two vectors, one must take into account that when the handedness of the coordinate system is *not* fixed a priori, the result is not a (true) vector but a pseudovector. Therefore, for consistency, the other side **must** also be a pseudovector.

More generally, the result of a cross product may be either a vector or a pseudovector, depending on the type of its operands (vectors or pseudovectors). Namely, vectors and pseudovectors are interrelated in the following ways under application of the cross product:

- vector × vector = pseudovector
- pseudovector × pseudovector = pseudovector
- vector × pseudovector = vector
- pseudovector × vector = vector.

So by the above relationships, the unit basis vectors **i**, **j** and **k** of an orthonormal, right-handed (Cartesian) coordinate frame **must** all be pseudovectors (if a basis of mixed vector types is disallowed, as it normally is) since **i** × **j** = **k**, **j** × **k** = **i** and **k** × **i** = **j**.

Because the cross product may also be a (true) vector, it may not change direction with a mirror image transformation. This happens, according to the above relationships, if one of the operands is a (true) vector and the other one is a pseudovector (e.g., the cross product of two vectors). For instance, a vector triple product involving three (true) vectors is a (true) vector.

A handedness-free approach is possible using exterior algebra.

10.9 Generalizations

There are several ways to generalize the cross product to the higher dimensions.

10.9.1 Lie algebra

Main article: Lie algebra

The cross product can be seen as one of the simplest Lie products, and is thus generalized by Lie algebras, which are axiomatized as binary products satisfying the axioms of multilinearity, skew-symmetry, and the Jacobi identity. Many Lie algebras exist, and their study is a major field of mathematics, called Lie theory.

For example, the Heisenberg algebra gives another Lie algebra structure on $\mathbf{R}^3$. In the basis $\{x,y,z\}$, the product is $[x,y]=z, [x,z]=[y,z]=0$.

10.9.2 Quaternions

Further information: quaternions and spatial rotation

The cross product can also be described in terms of quaternions, and this is why the letters **i**, **j**, **k** are a convention for the standard basis on $\mathbf{R}^3$. The unit vectors **i**, **j**, **k** correspond to "binary" (180 deg) rotations about their respective axes (Altmann, S. L., 1986, Ch. 12), said rotations being represented by "pure" quaternions (zero scalar part) with unit norms.

For instance, the above given cross product relations among **i**, **j**, and **k** agree with the multiplicative relations among the quaternions i, j, and k. In general, if a vector $[a_1, a_2, a_3]$ is represented as the quaternion $a_1i + a_2j + a_3k$, the cross product of two vectors can be obtained by taking their product as quaternions and deleting the real part of the result. The real part will be the negative of the dot product of the two vectors.

Alternatively, using the above identification of the 'purely imaginary' quaternions with $\mathbf{R}^3$, the cross product may be thought of as half of the commutator of two quaternions.

10.9.3 Octonions

See also: Seven-dimensional cross product and Octonion

A cross product for 7-dimensional vectors can be obtained in the same way by using the octonions instead of the quaternions. The nonexistence of nontrivial vector-valued cross products of two vectors in other dimensions is related to the result from Hurwitz's theorem that the only normed division algebras are the ones with dimension 1, 2, 4, and 8.

10.9.4 Wedge product

Main article: Exterior algebra

In general dimension, there is no direct analogue of the binary cross product that yields specifically a vector. There is however the wedge product, which has similar properties, except that the wedge product of two vectors is now a 2-vector instead of an ordinary vector. As mentioned above, the cross product can be interpreted as the wedge product in three dimensions by using the Hodge dual to map 2-vectors to vectors. The Hodge dual of the wedge product yields an $(n - 2)$-vector, which is a natural generalization of the cross product in any number of dimensions.

The wedge product and dot product can be combined (through summation) to form the geometric product.

10.9.5 Multilinear algebra

In the context of multilinear algebra, the cross product can be seen as the (1,2)-tensor (a mixed tensor, specifically a bilinear map) obtained from the 3-dimensional volume form,[note 2] a (0,3)-tensor, by raising an index.

In detail, the 3-dimensional volume form defines a product $V \times V \times V \to \mathbf{R}$, by taking the determinant of the matrix given by these 3 vectors. By duality, this is equivalent to a function $V \times V \to V^*$, (fixing any two inputs gives a function $V \to \mathbf{R}$ by evaluating on the third input) and in the presence of an inner product (such as the dot product; more generally, a non-degenerate bilinear form), we have an isomorphism $V \to V^*$, and thus this yields a map $V \times V \to V$, which is the cross product: a (0,3)-tensor (3 vector inputs, scalar output) has been transformed into a (1,2)-tensor (2 vector inputs, 1 vector output) by "raising an index".

Translating the above algebra into geometry, the function "volume of the parallelepiped defined by $(a, b, -)$ " (where the first two vectors are fixed and the last is an input), which defines a function $V \to \mathbf{R}$, can be *represented* uniquely as the dot product with a vector: this vector is the cross product $a \times b$. From this perspective, the cross product is *defined* by the scalar triple product, $\mathrm{Vol}(a, b, c) = (a \times b) \cdot c$.

In the same way, in higher dimensions one may define generalized cross products by raising indices of the n-dimensional volume form, which is a $(0, n)$ -tensor. The most direct generalizations of the cross product are to define either:

- a $(1,n-1)$ -tensor, which takes as input $n-1$ vectors, and gives as output 1 vector – an $(n-1)$ -ary vector-valued product, or
- a $(n-2,2)$ -tensor, which takes as input 2 vectors and gives as output skew-symmetric tensor of rank $n-2$ – a binary product with rank $n-2$ tensor values. One can also define $(k,n-k)$ -tensors for other k.

These products are all multilinear and skew-symmetric, and can be defined in terms of the determinant and parity.

The $(n-1)$ -ary product can be described as follows: given $n-1$ vectors $v_1,\dots,v_{n-1}$ in $\mathbf{R}^n$, define their generalized cross product $v_n=v_1\times\cdots\times v_{n-1}$ as:

- perpendicular to the hyperplane defined by the v_i,
- magnitude is the volume of the parallelotope defined by the v_i, which can be computed as the Gram determinant of the v_i,
- oriented so that $v_1,\dots,v_n$ is positively oriented.

This is the unique multilinear, alternating product which evaluates to $e_1\times\cdots\times e_{n-1}=e_n$, $e_2\times\cdots\times e_n=e_1$, and so forth for cyclic permutations of indices.

In coordinates, one can give a formula for this $(n-1)$ -ary analogue of the cross product in $\mathbf{R}^n$ by:

$$\bigwedge(\mathbf{v}_1,\dots,\mathbf{v}_{n-1})=\begin{vmatrix} {v_1}^1 & \cdots & {v_1}^n \\ \vdots & \ddots & \vdots \\ {v_{n-1}}^1 & \cdots & {v_{n-1}}^n \\ \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{vmatrix}.$$

This formula is identical in structure to the determinant formula for the normal cross product in $\mathbf{R}^3$ except that the row of basis vectors is the last row in the determinant rather than the first. The reason for this is to ensure that the ordered vectors ($\mathbf{v}_1$, ...,$\mathbf{v}_{n-1}$, Λ($\mathbf{v}_1$, ...,$\mathbf{v}_{n-1}$)) have a positive orientation with respect to ($\mathbf{e}_1$, ..., $\mathbf{e}_n$). If n is odd, this modification leaves the value unchanged, so this convention agrees with the normal definition of the binary product. In the case that n is even, however, the distinction must be kept. This $(n-1)$ -ary form enjoys many of the same properties as the vector cross product: it is alternating and linear in its arguments, it is perpendicular to each argument, and its magnitude gives the hypervolume of the region bounded by the arguments. And just like the vector cross product, it can be defined in a coordinate independent way as the Hodge dual of the wedge product of the arguments.

10.9.6 Skew-symmetric matrix

If the cross product is defined as a binary operation, it takes as *input* exactly two vectors. If its *output* is not required to be a vector or a pseudovector but instead a *matrix*, then it can be generalized in an arbitrary number of dimensions.[13][14][15]

In mechanics, for example, the angular velocity can be interpreted either as a pseudovector ω or as a anti-symmetric matrix or skew-symmetric tensor Ω . In the latter case, the velocity law for a rigid body looks:

$$\mathbf{v}_P-\mathbf{v}_Q=\Omega\cdot(\mathbf{r}_P-\mathbf{r}_Q)$$

where Ω is formally defined from the rotation matrix $R^{N\times N}$ associated to body's frame: $\Omega\triangleq\frac{dR}{dt}R^{\mathrm{T}}$. In three-dimensions holds:

$$\Omega=[\omega]_\times=\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

In quantum Mechanics the angular momentum L is often represented as an anti-symmetric matrix or tensor. More precisely, it is the result of cross product involving position $\mathbf{x}$ and linear momentum $\mathbf{p}$:

$L_{ij} = x_i p_j - p_i x_j$

Since both **x** and **p** can have an arbitrary number N of components, that kind of cross product can be extended to any dimension, holding the "physical" interpretation of the operation.

See § Alternative ways to compute the cross product for numerical details.

10.10 History

In 1773, the Italian mathematician Joseph Louis Lagrange, (born Giuseppe Luigi Lagrancia), introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions.[16] In 1843 the Irish mathematical physicist Sir William Rowan Hamilton introduced the quaternion product, and with it the terms "vector" and "scalar". Given two quaternions [0, **u**] and [0, **v**], where **u** and **v** are vectors in $\mathbf{R}^3$, their quaternion product can be summarized as [−**u** · **v**, **u** × **v**]. James Clerk Maxwell used Hamilton's quaternion tools to develop his famous electromagnetism equations, and for this and other reasons quaternions for a time were an essential part of physics education.

In 1878 William Kingdon Clifford published his Elements of Dynamic which was an advanced text for its time. He defined the product of two vectors[17] to have magnitude equal to the area of the parallelogram of which they are two sides, and direction perpendicular to their plane.

Oliver Heaviside in England and Josiah Willard Gibbs, a professor at Yale University in Connecticut, also felt that quaternion methods were too cumbersome, often requiring the scalar or vector part of a result to be extracted. Thus, about forty years after the quaternion product, the dot product and cross product were introduced—to heated opposition. Pivotal to (eventual) acceptance was the efficiency of the new approach, allowing Heaviside to reduce the equations of electromagnetism from Maxwell's original 20 to the four commonly seen today.[18]

Largely independent of this development, and largely unappreciated at the time, Hermann Grassmann created a geometric algebra not tied to dimension two or three, with the exterior product playing a central role. In 1853 Augustin-Louis Cauchy, a contemporary of Grassmann, published a paper on algebraic keys which were used to solve equations and had the same multiplication properties as the cross product.[19][20] William Kingdon Clifford combined the algebras of Hamilton and Grassmann to produce Clifford algebra, where in the case of three-dimensional vectors the bivector produced from two vectors dualizes to a vector, thus reproducing the cross product.

The cross notation and the name "cross product" began with Gibbs. Originally they appeared in privately published notes for his students in 1881 as *Elements of Vector Analysis*. The utility for mechanics was noted by Aleksandr Kotelnikov. Gibbs's notation and the name "cross product" later reached a wide audience through Vector Analysis, a textbook by Edwin Bidwell Wilson, a former student. Wilson rearranged material from Gibbs's lectures, together with material from publications by Heaviside, Föpps, and Hamilton. He divided vector analysis into three parts:

> First, that which concerns addition and the scalar and vector products of vectors. Second, that which concerns the differential and integral calculus in its relations to scalar and vector functions. Third, that which contains the theory of the linear vector function.

Two main kinds of vector multiplications were defined, and they were called as follows:

- The **direct**, **scalar**, or **dot** product of two vectors
- The **skew**, **vector**, or **cross** product of two vectors

Several kinds of triple products and products of more than three vectors were also examined. The above-mentioned triple product expansion was also included.

10.11 See also

- Bivector

- Cartesian product – A product of two sets
- Dot Product
- Exterior algebra
- Multiple cross products – Products involving more than three vectors
- Pseudovector
- × (the symbol)

10.12 Notes

[1] Here, "formal" means that this notation has the form of a determinant, but does not strictly adhere to the definition; it is a mnemonic used to remember the expansion of the cross product.

[2] By a volume form one means a function that takes in n vectors and gives out a scalar, the volume of the parallelotope defined by the vectors: $V\times\cdots\times V\to\mathbf{R}$. This is an n-ary multilinear skew-symmetric form. In the presence of a basis, such as on $\mathbf{R}^n$, this is given by the determinant, but in an abstract vector space, this is added structure. In terms of G-structures, a volume form is an SL -structure.

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Chapter 11

Cup product

In mathematics, specifically in algebraic topology, the **cup product** is a method of adjoining two cocycles of degree p and q to form a composite cocycle of degree $p + q$. This defines an associative (and distributive) graded commutative product operation in cohomology, turning the cohomology of a space X into a graded ring, $H^*(X)$, called the cohomology ring. The cup product was introduced in work of J. W. Alexander, Eduard Čech and Hassler Whitney from 1935–1938, and, in full generality, by Samuel Eilenberg in 1944.

11.1 Definition

In singular cohomology, the **cup product** is a construction giving a product on the graded cohomology ring $H^*(X)$ of a topological space X.

The construction starts with a product of cochains: if c^p is a p-cochain and d^q is a q-cochain, then

$$(c^p \smile d^q)(\sigma) = c^p(\sigma \circ \iota_{0,1,\ldots p}) \cdot d^q(\sigma \circ \iota_{p,p+1,\ldots,p+q})$$

where σ is a singular $(p + q)$ -simplex and $\iota_S, S \subset \{0, 1, ..., p+q\}$ is the canonical embedding of the simplex spanned by S into the $(p + q)$ -simplex whose vertices are indexed by $\{0, ..., p + q\}$.

Informally, $\sigma \circ \iota_{0,1,\ldots,p}$ is the p-th **front face** and $\sigma \circ \iota_{p,p+1,\ldots,p+q}$ is the q-th **back face** of σ, respectively.

The coboundary of the cup product of cocycles c^p and d^q is given by

$$\delta(c^p \smile d^q) = \delta c^p \smile d^q + (-1)^p(c^p \smile \delta d^q).$$

The cup product of two cocycles is again a cocycle, and the product of a coboundary with a cocycle (in either order) is a coboundary. The cup product operation induces a bilinear operation on cohomology,

$$H^p(X) \times H^q(X) \to H^{p+q}(X).$$

11.2 Properties

The cup product operation in cohomology satisfies the identity

$$\alpha^p \smile \beta^q = (-1)^{pq}(\beta^q \smile \alpha^p)$$

so that the corresponding multiplication is graded-commutative.

The cup product is functorial, in the following sense: if

$f: X \to Y$

is a continuous function, and

$f^*: H^*(Y) \to H^*(X)$

is the induced homomorphism in cohomology, then

$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta),$

for all classes α, β in $H^*(Y)$. In other words, f^* is a (graded) ring homomorphism.

11.3 Interpretation

It is possible to view the cup product $\smile: H^p(X) \times H^q(X) \to H^{p+q}(X)$ as induced from the following composition:

$C^\bullet(X) \times C^\bullet(X) \to C^\bullet(X \times X) \xrightarrow{\Delta^*} C^\bullet(X)$

in terms of the chain complexes of X and $X \times X$, where the first map is the Künneth map and the second is the map induced by the diagonal $\Delta: X \to X \times X$.

This composition passes to the quotient to give a well-defined map in terms of cohomology, this is the cup product. This approach explains the existence of a cup product for cohomology but not for homology: $\Delta: X \to X \times X$ induces a map $\Delta^*: H^\bullet(X \times X) \to H^\bullet(X)$ but would also induce a map $\Delta_*: H_\bullet(X) \to H_\bullet(X \times X)$, which goes the wrong way round to allow us to define a product. This is however of use in defining the cap product.

Bilinearity follows from this presentation of cup product, i.e. $(u_1 + u_2) \smile v = u_1 \smile v + u_2 \smile v$ and $u \smile (v_1 + v_2) = u \smile v_1 + u \smile v_2$.

11.4 Examples

Cup products may be used to distinguish manifolds from wedges of spaces with identical cohomology groups. The space $X := S^2 \vee S^1 \vee S^1$ has the same cohomology groups as the torus *T*, but with a different cup product. In the case of *X* the multiplication of the cochains associated to the copies of S^1 is degenerate, whereas in *T* multiplication in the first cohomology group can be used to decompose the torus as a 2-cell diagram, thus having product equal to **Z** (more generally *M* where this is the base module).

11.5 Other definitions

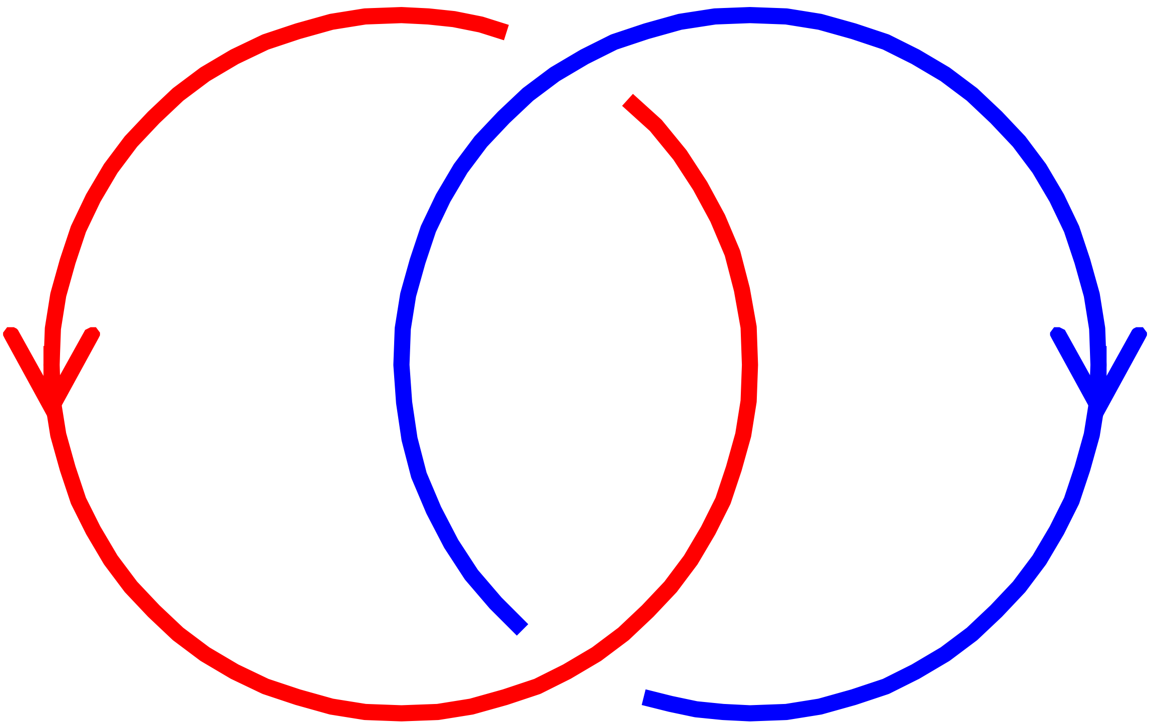
11.5.1 Cup product and differential forms

In de Rham cohomology, the cup product of differential forms is induced by the wedge product. In other words, the wedge product of two closed differential forms belongs to the de Rham class of the cup product of the two original de Rham classes.

11.5.2 Cup product and geometric intersections

When two submanifolds of a smooth manifold intersect transversely, their intersection is again a submanifold. By taking the fundamental homology class of these manifolds, this yields a bilinear product on homology. This product is dual to the cup product, i.e. the homology class of the intersection of two submanifolds is the Poincaré dual of the cup product of their Poincaré duals.

Similarly, the linking number can be defined in terms of intersections, shifting dimensions by 1, or alternatively in terms of a non-vanishing cup product on the complement of a link.



The linking number can be defined in terms of a non-vanishing cup product on the complement of a link. The complement of these two linked circles deformation retracts to a torus, which has a non-vanishing cup product.

11.6 Massey products

Main article: Massey product

The cup product is a binary (2-ary) operation; one can define a ternary (3-ary) and higher order operation called the Massey product, which generalizes the cup product. This is a higher order cohomology operation, which is only partly defined (only defined for some triples).

11.7 See also

- singular homology
- homology theory
- cap product
- Massey product
- Torelli group

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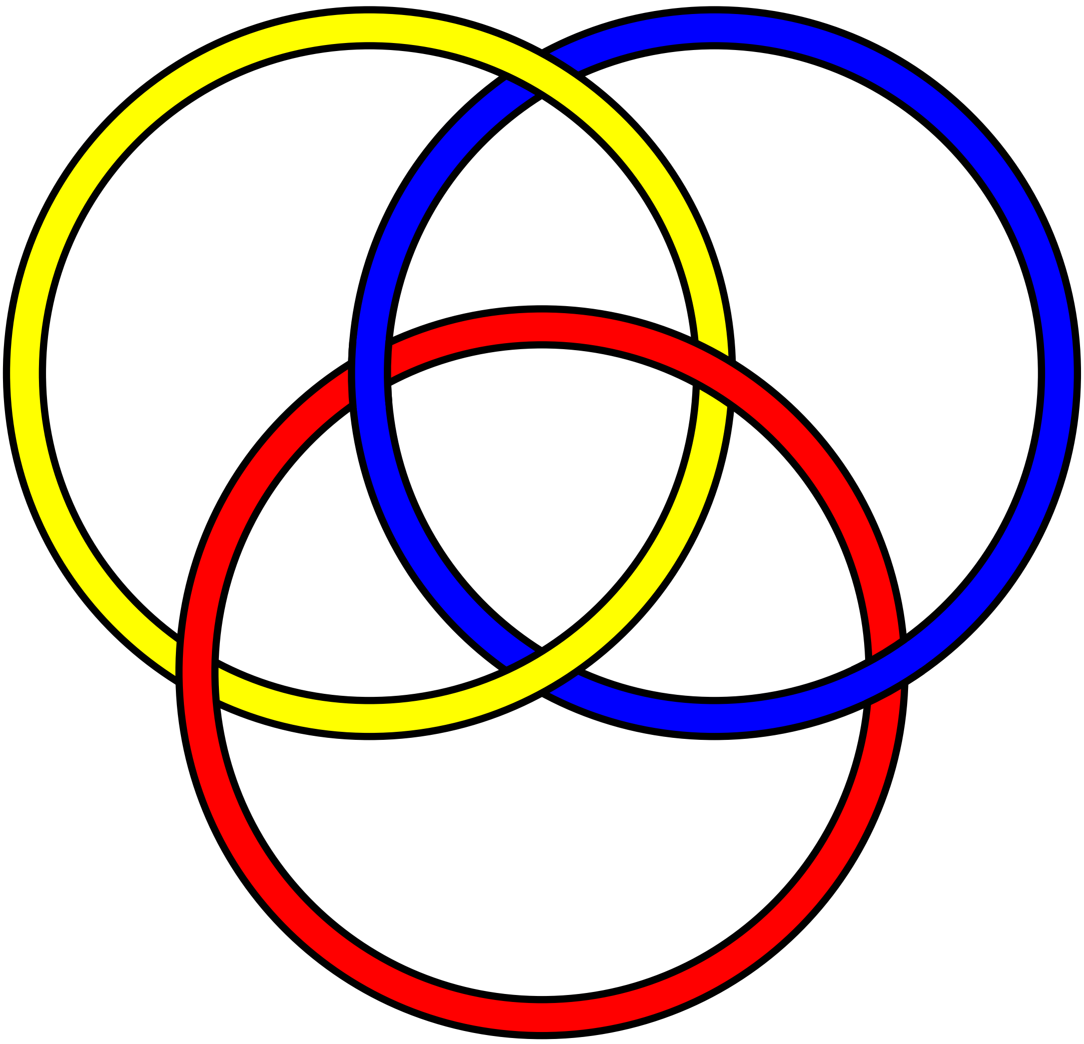

Massey products generalize cup product, allowing one to define "higher order linking numbers", the Milnor invariants.

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